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Hasse Carlsson

Michael Christ

Antonio Cordoba

Javier Duoandikoetxea

Jose L. Rudio de Francia

See next page for additional authors

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L^p ESTIMATES FOR MAXIMAL FUNCTIONS AND HILBERT TRANSFORMS ALONG FLAT CONVEX CURVES IN $\mathbf{R}^2$

BY HASSE CARLSSON[1], MICHAEL CHRIST[2], ANTONIO CORDOBA, JAVIER DUOANDIKOETXEA, JOSE L. RUBIO DE FRANCIA, JAMES VANCE, STEPHEN WAINGER[3], AND DAVID WEINBERG

1. Introduction and statement of results. Let $\Gamma: \mathbf{R} \to \mathbf{R}^n$ be a curve in $\mathbf{R}^n$ with $\Gamma(0) = 0$. For suitable test functions f, let $H_\Gamma f(x) = p.v. \int_{-a}^{a} f(x - \Gamma(t))t^{-1}\,dt$ and $M_\Gamma f(x) = \sup_{0<r\le 1} |r^{-1} \int_0^r f(x - \Gamma(t))\,dt|$. H_Γ and M_Γ are called the Hilbert transform and maximal function along Γ, respectively. There has been considerable interest in estimates of the form $||H_\Gamma f||_p \le C||f||_p$ and $||M_\Gamma f||_p \le C||f||_p$ where $||\cdot||_p$ denotes the norm in $L^p(\mathbf{R}^n)$.

If Γ has some curvature at the origin, in a weak sense, then the above L^p estimates for H_Γ and M_Γ have been proved for $1 < p < \infty$ and $1 < p \le \infty$ respectively, via techniques developed by Nagel, Riviere, Stein, and Wainger; see the survey [**SW**] and the references given there. More recently there has been interest in the case when Γ is flat to infinite order at $t = 0$. In particular if $\Gamma(t) = (t, \gamma(t))$ is a curve in $\mathbf{R}^2$ for which γ is convex for $t > 0$ and either even or odd, then a necessary and sufficient condition for H_Γ to be bounded on L^2 has been obtained in [**NVWW1**]. The condition for odd γ has also turned out to imply the L^2 boundedness of M_Γ [**NVWW2**]. There has also been progress in the study of L^p boundedness for $p \ne 2$ [**NW, CNVWW, C**].

In the present paper we consider (locally) C^1 curves $\Gamma(t) = (t, \gamma(t))$ in $\mathbf{R}^2$ defined for $t \ge 0$, with $\gamma'(0) = \gamma(0) = 0$, convex and increasing. To discuss the Hilbert transform $\Gamma(t)$ must be defined for $t < 0$; we define $\Gamma_e(t) = (t, \gamma(-t))$ and $\Gamma_0(t) = (t, -\gamma(-t))$ for $t < 0$. Curvature hypotheses are replaced by the much weaker "doubling property"

(1.1) there exists $\lambda > 1$ with $\gamma'(\lambda t) \ge 2\gamma'(t)$ for all $t > 0$.

We shall prove

THEOREM. *Let $\Gamma, \Gamma_e, \Gamma_0$ be as above and satisfy* (1.1). *Then $||M_\Gamma f||_p \le C||f||_p$ for $1 < p \le \infty$, and $||H_{\Gamma_e} f||_p + ||H_{\Gamma_0} f||_p \le C||f||_p$ for $1 < p < \infty$. More precisely, the latter assertion is that the operators H_Γ, initially defined only for test functions, extend to bounded operators on L^p.*

By combining this theorem with the necessary condition for L^2 boundedness of H_{Γ_e} in [**NVWW1**], we obtain the following

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COROLLARY. *For all curves Γ_e as above, and for all p, $1 < p < \infty$, a necessary and sufficient condition for the boundedness of H_{Γ_e} on L^p is* (1,1).

(In fact, we can see that H_{Γ_e} is not even of weak type (p,p) for any p, unless (1.1) holds: for $0 < a < A$, let S be the quadrilateral with vertices at $(\pm a, 0)$, $(-2A, \gamma'(A)(-2A-a))$, $(-2A, \gamma'(a)(-2A+a))$; let T have vertices at $(0,0), (a,0)$ $(-A, -A\gamma'(A))$, $(a-A, -A\gamma'(A))$; then $H_{\Gamma_e}(\chi_S) > \log(A/a)$ on T, since Γ_e is even and convex. But, denying (1.1) implies that $|S|/|T|$ can be bounded while $A/a \to \infty$.)

In previous work proofs of L^p estimates of the type under discussion here have depended upon favorable decay estimates for Fourier transforms of certain measures supported on the curve Γ. In limiting cases in which Γ consists of an infinite sequence of line segments tending to the origin such estimates fail to hold, yet (1.1) may be satisfied. The principal innovation here is a Littlewood-Paley argument based on a decomposition of the Fourier transform plane into lacunary sectors as in [**NSW**]. A preliminary result based on this technique was proved in [**CNVWW**]. A similar idea was also previously used in [**NSW**] in studying the "lacunary" maximal function. Subsequently [**DRdF**] showed how old results, for cases in which favorable decay estimates do hold, could be proved by clever applications of classical Littlewood-Paley decompositions. A combination of these ideas leads to the proof of the theorem in this paper.

2. A Paley-Littlewood decomposition. Now we describe a Paley-Littlewood decomposition. Let $\alpha_k = \gamma'(\lambda^k)$. Then by using the Marcinkiewicz multiplier theorem, (1.1), duality, and standard techniques, we can find multiplier operators P_k defined by $(P_k f)\widehat{\ }(\xi,\eta) = \Phi_k(\xi,\eta)\cdot \hat f(\xi,\eta)$ such that

$$\sum_k P_k = \text{identity};$$

$$\operatorname{supp}\Phi_k \subseteq \{(\xi,\eta)\colon \alpha_{k-2} < |\xi/\eta| < \alpha_{k+1}\};$$

$$\left\|\left(\sum_k |P_k f|^2\right)^{1/2}\right\|_p \le C_p\|f\|_p, \qquad 1 < p < \infty;$$

and

$$\left\|\sum_k P_k f_k\right\|_p \le C_p \left\|\left(\sum_k |P_k f_k|^2\right)^{1/2}\right\|_p, \qquad 1 < p < \infty.$$

3. The proof of $\|M_\Gamma f\|_p \le C\|f\|_p$ for $1 < p \le \infty$. We may assume $\lambda \ge 2$. For each integer k let I_k be the interval $[\lambda^{k-1}, \lambda^k]$. Define measures μ_k by their action on test functions ϕ: $\mu_k(\phi) = |I_k|^{-1}\int_{I_k}\phi(t,\gamma(t))\,dt$. Then

$$(\mu_k)\widehat{\ }(\xi,\eta) = |I_k|^{-1}\int_{I_k}\exp(i\xi t + i\eta\gamma(t))\,dt.$$

The L^p boundedness of M_Γ is equivalent to

$$\|\sup_k|\mu_k * f|\,\|_p \le C\|f\|_p, \qquad 1 < p \le \infty. \tag{3.1}$$

The proof of 3.1 will be by a bootstrapping argument similar to that of [**NSW**]. We prove the following two lemmas:

LEMMA 1. $\|\sup_k |\mu_k * f|\,\|_2 \le C\|f\|_2$. *Moreover, if there exists* $r < 2$ *and* $C < \infty$ *with*

$$\left\|\left(\sum_k |\mu_k * f_k|^2\right)^{1/2}\right\|_r \le C\left\|\left(\sum_k |f_k|^2\right)^{1/2}\right\|_r \tag{3.2}$$

for all sequences f_k, *then for each* $r < p \le 2$ *there exists* $C_p < \infty$ *such that* $\|\sup_k |\mu_k * f|\,\|_p \le C_p\|f\|_p$.

LEMMA 2. *If* $\|\sup_k |\mu_k * f|\,\|_p \le C_p\|f\|_p$ *for some* p, $1 < p \le 2$, *then* $\|(\sum_k |\mu_k * f_k|^2)^{1/2}\|_r \le C\|(\sum_k |f_k|^2)^{1/2}\|_r$ *for all* r *with* $r^{-1} < (1+p^{-1})/2$.

3.1 follows by applying Lemmas 1 and 2 infinitely often as in [**NSW**]. The proof of Lemma 2 is the same as the proof of Lemma 3 of [**NSW**].

To prove Lemma 1 we compare μ_k to σ_k where $\sigma_k = \mu_k * [(\phi_k - \delta)\otimes(\psi_k - \delta)]$. Here $\phi(t)$, $\psi(t)$ are nonnegative C^∞ functions on $\mathbf{R}$ with support in $[-1,1]$ and $\int\phi = \int\psi = 1$; $\phi_k(t) = \lambda^{-k}\phi(\lambda^{-k}t)$, and

$$\psi_k(t) = [\gamma(\lambda^{k+1})]^{-1}\psi[(\gamma(\lambda^{k+1}))^{-1}t].$$

δ is the dirac point mass at the origin. The meaning of $(\phi_k - \delta)\otimes(\psi_k - \delta)$ is that $\phi_k - \delta$ acts on the first variable and $\psi_k - \delta$ on the second. We set $\nu_k = \mu_k - \sigma_k$. Notice that

$$\nu_k = \mu_k * (\phi_k \otimes \delta) + \mu_k * (\delta \otimes \psi_k) - \mu_k * (\phi_k \otimes \psi_k)$$

is a sum of smoothed out μ_k. One can show $\sup_k |\nu_k * f|(x,y) \le CM_s f(x,y)$ where M_s is the usual strong maximal function. Thus,

$$\|\sup_k |\nu_k * f|\,\|_p \le C_p\|f\|_p, \tag{3.3}$$

$$\left\|\left(\sum_k |\nu_k * f_k|^2\right)^{1/2}\right\|_p \le C_p\left\|\left(\sum_k |f_k|^2\right)^{1/2}\right\|_p \tag{3.4}$$

both hold for $1 < p \le \infty$; see [**FS**].

To prove Lemma 1 it suffices to bound $\sup_k |\sigma_k * f|$, in view of 3.3. But (letting P_k be as in §2)

$$\begin{aligned}\sup_k |\sigma_k * f| &= \sup_k \left|\sum_j \sigma_k * P_{j+k} f\right| \\ &\le \sum_j \sup_k |\sigma_k * P_{j+k} f| \\ &\le \sum_j \left(\sum_k |\sigma_k * P_{j+k} f|^2\right)^{1/2} \equiv \sum_j G_j f.\end{aligned}$$

We show

$$(3.5) \qquad ||G_j f||_p \le C||f||_p, \qquad r < p \le 2;$$

$$(3.6) \qquad ||G_j f||_2 \le C \cdot 2^{-|j|/2}||f||_2.$$

3.5 and 3.6 imply the conclusion of Lemma 1 by a standard interpolation argument. 3.5 follows from §2, 3.2, and 3.4, 3.6 follows from the following estimates on $\hat{\sigma}_k(\xi,\eta)$: $|\hat{\sigma}_k(\xi,\eta)| \le C\lambda^k|\xi|$, $|\hat{\sigma}_k(\xi,\eta)| \le C\gamma(\lambda^{k+1})|\eta|$, and $|\hat{\sigma}_k(\xi,\eta)| \le |I_k|^{-1} \max_{t\in I_k} |\xi + \eta\gamma'(t)|^{-1}$.

4. The proof of $||H_\Gamma f||_p \le C_p||f||_p$, $1 < p < \infty$**.** The proof is similar to the proof in §3. The analogue of the operation $f \to \sigma_k * f$ is

$$L_k f = H_k\{[(\phi_k - \delta) \otimes (\psi_k - \delta)] * f\},$$

where $H_k g(x,y) = \int_{|t|\in I_k} g(x-t, y-\gamma(t))t^{-1}\,dt$. Then we must show

$$\left|\left|\sum_k P_{j+k}L_k f\right|\right|_p \le C||f||_p \quad \text{and} \quad \left|\left|\sum_k P_{j+k}L_k f\right|\right|_2 \le C \cdot 2^{-|j|/2}||f||_2.$$

The latter follows from simple Fourier transform estimates. For the former,

$$\begin{aligned}
\left|\left|\sum_k P_{j+k}L_k f\right|\right|_p &\le C\left|\left|\left(\sum_k |P_{j+k}L_k f|^2\right)^{1/2}\right|\right|_p \\
&= C\left|\left|\left\{\sum_k |\mu_k * [(\phi_k - \delta)\otimes(\psi_k - \delta)] * P_{j+k} f|^2\right\}^{1/2}\right|\right|_p \\
&\le C\left|\left|\left\{\sum_k |[(\phi_k - \delta)\otimes(\psi_k - \delta)] * P_{j+k} f|^2\right\}^{1/2}\right|\right|_p \\
&\le C\left|\left|\left\{\sum_k |P_{j+k} f|^2\right\}^{1/2}\right|\right|_p \le C||f||_p,
\end{aligned}$$

by §2, Lemmas 1 and 2, and [**FS**].

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DEPARTMENT OF MATHEMATICS, CHALMERS UNIVERSITY OF TECHNOLOGY, S-412 96 GOTEBÖRG, SWEDEN

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08544

UNIVERSIDAD AUTÓNOMA DE MADRID, FACULTAD DE CIENCIAS, SECCIÓN DE MATEMÁTICAS, MADRID 34, SPAIN (Current address of Antonio Cordoba, Javier Duoandikoetxea and José L. Rubio de Francia)

DEPARTMENT OF MATHEMATICS AND STATISTICS, WRIGHT STATE UNIVERSITY, DAYTON, OHIO 45435

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MADISON, MADISON, WISCONSIN 53706

DEPARTMENT OF MATHEMATICS, TEXAS TECH UNIVERSITY, LUBBOCK, TEXAS 79409