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Hongxun Qin

Dan Slilaty
Wright State University - Main Campus, daniel.slilaty@wright.edu

Xiangqian Zhou
Wright State University - Main Campus, xiangqian.zhou@wright.edu

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The Regular Excluded Minors for Signed-Graphic Matroids

HONGXUN QIN¹, DANIEL C. SLILATY²† and XIANQQIAN ZHOU²

¹Department of Mathematics, The George Washington University, Washington DC, 20052, USA
(e-mail: hqin@gwu.edu)

²Department of Mathematics and Statistics, Wright State University, Dayton, OH 45435, USA
(e-mail: {daniel.slilaty|xianqian.zhou}@wright.edu)

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Dedicated to Thomas Dowling on the occasion of his retirement

We show that the complete list of regular excluded minors for the class of signed-graphic matroids is $M^*(G_1),\ldots,M^*(G_{29}),R_{15},R_{16}$. Here $G_1,\ldots,G_{29}$ are the vertically 2-connected excluded minors for the class of projective-planar graphs and $R_{15}$ and $R_{16}$ are two regular matroids that we will define in the article.

1. Introduction

We assume the reader is familiar with matroid theory as in [7]. If the reader is not familiar with signed graphs and their matroids as in [19], then we review all of the relevant material in Section 2. Signed-graphic matroids are exactly the minors of Dowling geometries [3] for the group of order two. Our main result is Theorem 1.1. Here $G_1,\ldots,G_{29}$ are the vertically 2-connected excluded minors for the class of projective-planar graphs.¹ The matroids $R_{15}$ and $R_{16}$ are introduced in Section 4.

Theorem 1.1. A regular matroid $M$ is a signed-graphic matroid if and only if $M$ contains none of the following as a minor: $M^*(G_1),\ldots,M^*(G_{29}),R_{15},$ and $R_{16}$.

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¹ There are 35 excluded minors for the class of projective-planar graphs and 29 of these are vertically 2-connected. See Archdeacon [1] and Glover, Huneke and Wang [4], or see [6, Theorem 6.5.1].
Whittle conjectures in [18] that there is a theorem for near-regular matroids similar to Theorem 1.2 that uses signed-graphic matroids and co-signed-graphic matroids as the basic terms in the decomposition. Since the proof of Theorem 1.2 uses the list of excluded minors for the class of graphic matroids, it is possible that a result for near-regular matroids would use the list of excluded minors for the class of signed-graphic matroids.

**Theorem 1.2 (Seymour [10]).** Every regular matroid $M$ is constructed by a sequence of $k$-sums ($k \in \{1, 2, 3\}$) of graphic matroids, cographic matroids, and copies of the matroid $R_{10}$.

Since we are working within the class of regular matroids, it is no surprise that a proof of Theorem 1.1 starts with Theorem 1.2. Given Theorem 1.2, we know that if $M$ is a regular excluded minor for the class of signed-graphic matroids and $M$ is internally 4-connected, then $M$ is either graphic, cographic, or $R_{10}$. Since a graphic matroid is signed-graphic and since $R_{10}$ is the matroid of the signed graph $-K_5$ (by $-G$ we mean the signed graph with $G$ as its underlying graph and all edges signed negatively) we must have that $M$ is cographic. By part (1) of Theorem 1.3, we then get that $M \in \{M^*(G_1), \ldots, M^*(G_{29})\}$. Also, part (2) of Theorem 1.3 tells us that each of $M^*(G_1), \ldots, M^*(G_{29})$ is indeed an excluded minor for the class of signed-graphic matroids.

**Theorem 1.3.** Let $M$ be a cographic matroid.

1. If $M$ is 3-connected and an excluded minor for the class of signed-graphic matroids, then $M \in \{M^*(G_1), \ldots, M^*(G_{29})\}$.
2. If $M \in \{M^*(G_1), \ldots, M^*(G_{29})\}$, then $M$ is an excluded minor for the class of signed-graphic matroids.

**Proof.** Write $M = M^*(G)$, where $G$ has no isolated vertices.

1. Since $M/e = M^*(G \setminus e)$ and $M \setminus e = M^*(G/e)$ are both signed-graphic and connected, Theorem 1.4 implies that $G/e$ and $G \setminus e$ are both projective-planar, while $G$ is not projective-planar, and so $G \in \{G_1, \ldots, G_{29}\}$.

2. Since $M(G)$ is connected, $G/e$ and $G \setminus e$ are both connected, and so Theorem 1.4 implies that $M^*(G)/e = M^*(G \setminus e)$ and $M^*(G) \setminus e = M^*(G/e)$ are both signed-graphic, while $M(G)$ is not.

**Theorem 1.4.** Let $G$ be a connected graph.

1. If $G$ is projective-planar, then $M^*(G)$ is signed-graphic [11].
2. If $M^*(G)$ is connected and signed-graphic, then $G$ is projective-planar [8, 12].

Therefore, our work in this paper is to show that if $M$ has connectivity $k \in \{2, 3\}$, then $M \in \{M^*(G_1), \ldots, M^*(G_{29}), R_{15}, R_{16}\}$ and that $R_{15}$ and $R_{16}$ are excluded minors. The case for $k = 2$ is done in Section 3 and the case for $k = 3$ is done in Section 6. In Section 2 we have some preliminaries, in Section 4 we introduce the matroids $R_{15}$ and $R_{16}$, and in Section 5 we present some lemmas that we will use in Section 6.
2. Preliminaries

Graphs. A graph $G$ consists of a collection of vertices (i.e., topological 0-cells), denoted by $V(G)$, and a set of edges (i.e., topological 1-cells), denoted by $E(G)$, where an edge has two ends, each of which is attached to a vertex. A link is an edge that has its ends incident to distinct vertices and a loop is an edge that has both of its ends incident to the same vertex.

A circle is a connected, 2-regular graph (i.e., a simple closed path). In graph theory a circle is often called a cycle, circuit, polygon, etc. We denote the cycle matroid of the graph $G$ by $M(G)$. If $X \subseteq E(G)$, then we denote the subgraph of $G$ consisting of the edges in $X$ and all vertices incident to an edge in $X$ by $G\! :X$. The collection of vertices in $G\! :X$ is denoted by $V(X)$, the number of vertices in $G\! :X$ is denoted by $v_X$, and the number of connected components in $G\! :X$ is denoted by $c_X$.

For $k \geq 1$, a $k$-separation of a graph is a bipartition $(A,B)$ of the edges of $G$ such that $|A| \geq k$, $|B| \geq k$, and $|V(A) \cap V(B)| = k$. A vertical $k$-separation $(A,B)$ of $G$ is a $k$-separation where $V(A) \setminus V(B) \neq \emptyset$ and $V(B) \setminus V(A) \neq \emptyset$. A separation or vertical separation $(A,B)$ is said to have connected parts when $G\! :A$ and $G\! :B$ are both connected. A connected graph on at least $k+1$ vertices is said to be vertically $k$-connected when there is no vertical $r$-separation for $r < k$. Vertical $k$-connectivity is usually called $k$-connectivity, but here we wish to distinguish between this kind of graph connectivity and the second type used in Tutte’s book on graph theory [17].

Given a subgraph $H$ of $G$, an $H$-bridge is either an edge not in $H$ whose end-points are both in $H$ or a connected component $C$ of $G \setminus V(H)$ along with the links between $C$ and $H$. Given an $H$-bridge $B$ of $G$: a foot of $B$ is an edge of $B$ with an end-point in $H$, a vertex of attachment of $B$ is a vertex in $H$ that is an end-point of a foot of $B$, and $\overline{B}$ denotes the bridge $B$ minus the vertices of attachment of $B$ (i.e., either a connected component of $G \setminus V(H)$ or $\emptyset$ when $B$ is a single edge). An $H$-bridge of $G$ with $n$ vertices of attachment is called an $n$-bridge.

If $G'$ is a subdivision of a graph $G$ where $G$ has minimum degree at least three, then a branch vertex of $G'$ is a vertex of degree at least three in $G'$ and a branch is a path in $G'$ corresponding to an edge in $G$. A $G'$-bridge $B$ is called local if all attachments of $B$ are on the same branch of $G'$. A useful fact about local bridges that we will need later is Proposition 2.1.

Proposition 2.1 ([6, Lemma 6.2.1]). Let $G$ be a vertically 3-connected graph. If $H \subseteq G$ is a subdivision of a graph $\Gamma$, then there is a subdivision $H'$ of $\Gamma$ in $G$ such that $H'$ has the same branch vertices as $H$, if $e$ is a branch in $H$ then the corresponding branch $e'$ in $H'$ connects the same branch vertices, and $H'$ has no local bridges.

Signed graphs. A signed graph is a pair $(G,\sigma)$ in which $\sigma: E(G) \to \{+1,-1\}$. A circle or path in a signed graph $\Sigma$ is called positive if the product of signs on its edges is positive, otherwise the circle or path is called negative. If $H$ is a subgraph of $\Sigma$, then $H$ is called balanced when all circles in $H$ are positive. A balancing vertex of an unbalanced signed graph is a vertex whose removal leaves a balanced subgraph.
A switching function on a signed graph $\Sigma = (G, \sigma)$ is a function $\eta : V(\Sigma) \to \{+1,-1\}$. The signed graph $\Sigma^\eta = (G, \sigma^\eta)$ has sign function $\sigma^\eta$ defined on all edges of $G$ by $\sigma^\eta(e) = \eta(v)\sigma(e)\eta(w)$, where $v$ and $w$ are the end-points of $e$. When two signed graphs $\Sigma_1$ and $\Sigma_2$ satisfy $\Sigma_1^\eta = \Sigma_2$ for some switching function $\eta$, $\Sigma_1$ and $\Sigma_2$ are said to be switching-equivalent. Two signed graphs with the same underlying graph are switching-equivalent if and only if they have the same list of positive circles (see [19, Proposition 3.2]). Switching-equivalent signed graphs are considered to be isomorphic.

In a signed graph $\Sigma = (G, \sigma)$, the deletion of $e$ from $\Sigma$ is defined as $\Sigma \setminus e = (G \setminus e, \sigma)$, where $\sigma$ is restricted to the domain $E(G \setminus e)$. The contraction of an edge $e$ is defined for three distinct cases. If $e$ is a link, then $\Sigma/e = (G/e, \sigma^\eta)$, where $\eta$ is a switching function on $\Sigma$ satisfying $\sigma^\eta(e) = +1$. Of course, then $\sigma^\eta$ is restricted to the edges of $G/e$ in $\Sigma/e$. Note that $\Sigma/e$ is well defined up to switching. If $e$ is a positive loop, then $\Sigma/e = \Sigma \setminus e$. If $e$ is a negative loop incident with vertex $v$, then $\Sigma/e$ is the signed graph obtained from $\Sigma$ as follows: links incident to $v$ become negative loops incident to their other end-point, negative loops incident to $v$ other than $e$ become positive loops incident to $v$, and edges not incident to $v$ remain unchanged. The reason for this definition of contraction in signed graphs is so that contractions in signed graphs will correspond to contractions in their signed-graphic matroids.

A minor of $\Sigma$ is a signed graph obtained from $\Sigma$ by a sequence of contractions and deletions of edges, deletions of isolated vertices, and switchings. A link minor is a minor that is obtained without contracting any negative loops.

A signed graph is called tangled if it is unbalanced, has no balancing vertex, and no two vertex-disjoint negative circles. The proof of Proposition 2.2 is straightforward and is left to the reader.

**Proposition 2.2.** If $\Sigma$ is tangled, then $\Sigma$ has exactly one unbalanced block; in particular, $\Sigma$ has no negative loops.

**Proposition 2.3.** If $\Sigma_1$ and $\Sigma_2$ are tangled, $\Sigma_1$ is a minor of $\Upsilon$, and $\Upsilon$ is a minor of $\Sigma_2$, then $\Upsilon$ is tangled and is a link minor of $\Sigma_2$.

**Proof.** Let $\mathcal{B}$ be the class of balanced signed graphs, let $\mathcal{J}$ be the class of signed graphs that are balanced after removing negative loops, let $\mathcal{V}$ be the class of signed graphs with balancing vertices, and let $\mathcal{T}$ be the class of tangled signed graphs. By the definitions of these types of signed graphs and the definition of contractions in signed graphs we get the following three facts: since tangled signed graphs do not have negative loops (Proposition 2.2), any 1-edge deletion or contraction of a member of $\mathcal{T}$ is in $\mathcal{T}$ or $\mathcal{V}$; any 1-edge deletion or contraction of a member of $\mathcal{V}$ is in $\mathcal{T}$, $\mathcal{J}$, or $\mathcal{B}$; and any 1-edge deletion or contraction of a member of $\mathcal{J}$ or $\mathcal{B}$ is in $\mathcal{J}$ or $\mathcal{B}$. Hence, when obtaining a tangled minor of a tangled signed graph, we contract only links and never leave the class of tangled signed graphs. \qed
When drawing signed graphs, positive edges are drawn as solid curves and negative edges as dashed curves. A signed graph is said to be vertically $k$-connected when its underlying graph is vertically $k$-connected.

**Signed-graphic matroids.** The frame matroid (often called the bias matroid) of $\Sigma$ is denoted by $M(\Sigma)$. In this paper such a matroid is simply called a signed-graphic matroid. The element set of $M(\Sigma)$ is $E(\Sigma)$ and a circuit of $M(\Sigma)$ is either the edge set of a positive circle or the edge set of a subdivision of a subgraph in Figure 1 with no positive circles.

For any $e \in E(\Sigma)$, we have that $M(\Sigma \setminus e) = M(\Sigma) \setminus e$ and $M(\Sigma/e) = M(\Sigma)/e$ (see [19, Theorem 5.2]). Note that if $\Sigma = (G, \sigma)$ is balanced, then $M(\Sigma) = M(G)$. Hence, the class of signed-graphic matroids contains the class of graphic matroids. Given two signed graphs $\Sigma_1$ and $\Sigma_2$ with the same underlying graph, $M(\Sigma_1) = M(\Sigma_2)$ if and only if $\Sigma_1$ and $\Sigma_2$ have the same positive and negative circles, which holds if and only if $\Sigma_1$ and $\Sigma_2$ are switching-equivalent.

Given $X \subseteq E(\Sigma)$, we denote the number of balanced connected components of $\Sigma : X$ by $b_X$. If $X \subseteq E(\Sigma)$, then $r(X) = v_X - b_X$ (see [19, Theorem 5.1(j)]). For brevity we write $r(\Sigma)$ to mean $r(M(\Sigma))$. The rank function tells us that if $\Sigma$ is not connected after removing isolated vertices, then $M(\Sigma)$ is not connected. It also tells us that a cocircuit of $M(\Sigma)$ is a minimal set of edges whose removal increases the number of balanced components by one.

Theorem 2.4 is from [14, Theorems 1.3 and 1.4]. It tells us that regularity of $M(\Sigma)$ is almost synonymous with $\Sigma$ being tangled. Theorem 2.5 is an important fact relating matroid connectivity and graph connectivity.

**Theorem 2.4 (Slilaty and Qin [14]).** If $\Sigma$ is connected, then the following are true.

1. If $\Sigma$ is tangled, then $M(\Sigma)$ is regular.
2. If $M(\Sigma)$ is regular and not graphic, then $\Sigma$ is tangled.

**Theorem 2.5 (Slilaty and Qin [15, Theorem 1.6]).** If $\Sigma$ is tangled and has no isolated vertices and $M(\Sigma)$ is $k$-connected for any $k \in \{2, 3\}$, then $\Sigma$ is vertically $k$-connected.

If $\Sigma$ is a signed graph with balancing vertex $v$, then by switching we may assume that all negative edges of $\Sigma$ are incident to $v$. Let $G_v$ be the graph obtained from $\Sigma$ by splitting $v$ into two vertices $v_+$ and $v_-$, where positive links incident to $v$ become links incident to
v_+, negative links incident to v become links incident to v_-, negative loops incident to v become \(v_+v_-\)-links, and positive loops incident to v are positive loops anywhere in \(G_v\).

Proposition 2.6. If \(\Sigma\) has a balancing vertex \(v\) and \(G_v\) is the graph obtained from \(\Sigma\) as in the previous paragraph, then \(M(\Sigma) = M(G_v)\).

1-sums. Let \(\Sigma\) and \(\Upsilon\) be signed graphs with non-empty edge sets such that \(\Upsilon\) is balanced. The 1-sum of \(\Sigma\) and \(\Upsilon\) is the identification of \(\Sigma\) and \(\Upsilon\) along some vertex and is denoted by \(\Sigma \oplus_1 \Upsilon\). Proposition 2.7 is immediate from our definition of a signed-graphic 1-sum and the definition of a matroid 1-sum.

Proposition 2.7. If \(\Sigma\) and \(\Upsilon\) are signed graphs, then \(M(\Sigma \oplus_1 \Upsilon) = M(\Sigma) \oplus_1 M(\Upsilon)\).

2-sums. Given two signed graphs \(\Sigma\) and \(\Upsilon\), we will define two methods of taking their 2-sum. By \(\Sigma \oplus_2 \Upsilon\) we mean a 2-sum that is one of these two types. If both of \(\Sigma\) and \(\Upsilon\) are unbalanced, then the 1-vertex 2-sum is obtained by identifying the signed graphs along a negative loop and then deleting the negative loop. If exactly one of \(\Sigma\) and \(\Upsilon\) is unbalanced, then the 2-vertex 2-sum of the signed graphs is obtained by choosing a link in each signed graph, switching so that the links have the same sign in each, identifying the two signed graphs along the links, and then deleting that link. In both cases it is required that the edge along which the 2-sum is taken is not a coloop in the signed-graphic matroid. The verification of Proposition 2.8 is routine.

Proposition 2.8. If \(\Sigma\) and \(\Upsilon\) are signed graphs, then \(M(\Sigma \oplus_2 \Upsilon) = M(\Sigma) \oplus_2 M(\Upsilon)\).

Proposition 2.9. If \(M_1\) is a signed-graphic matroid and \(M_2\) is a graphic matroid, then \(M_1 \oplus_2 M_2\) is signed-graphic.

Proof. Say that \(M_1 = M(\Sigma)\), \(M_2 = M(G)\), and \(e\) is the edge in each of \(\Sigma\) and \(G\) along which the 2-sum is taken. Since \(e\) is not a matroid loop, \(e\) is a link in \(G\) (call its end-points \(v\) and \(w\)). If \(e\) is a link in \(\Sigma\), then let \(\Upsilon\) be the signed graph with underlying graph \(G\) and all edges signed positively. Note that \(M(\Upsilon) = M(G)\). If \(e\) is a negative loop in \(\Sigma\), then let \(\Upsilon\) be the signed graph with a balancing vertex obtained from \(G\) by the reverse of the operation described in Proposition 2.6 performed on the end-points of \(e\) in \(G\). Note that \(e\) is then a negative loop in \(\Upsilon\). In either case Proposition 2.8 implies that \(M_1 \oplus_2 M_2 = M(\Sigma) \oplus_2 M(\Upsilon) = M(\Sigma \oplus_2 \Upsilon)\), as required.

3-sums. Given a signed graph \(\Sigma\) and a balanced signed graph \(\Upsilon\) (or a graph \(G\)), their 3-vertex 3-sum is obtained by selecting a positive triangle in each term, switching so that the edges of the triangle have the same sign pattern in each term, identifying the signed graphs along the triangles, and then deleting the edges.

We also make use of the operation of symmetric differences of binary matroids. If \(M_1\) and \(M_2\) are binary matroids on edge sets \(E_1\) and \(E_2\) with \(E_1 \cap E_2 \neq \emptyset\), then there is a
binary matroid $M_1 \triangle M_2$ on edge set $E_1 \triangle E_2$ whose circuits are the minimal non-empty elements of $\{C_1 \triangle C_2 : C_i$ is a (possibly empty) disjoint union of circuits in $M_i\}$ that are contained in $E_1 \triangle E_2$. An important property of this symmetric difference operation is that $(M_1 \triangle M_2)^* = M_1^* \triangle M_2^*$ [10, p. 319]. In [10], the 3-sum $M_1 \oplus_3 M_2$ for binary matroids is defined as $M_1 \triangle M_2$, where $E_1 \cap E_2$ is a triangle in each $M_i$ that is co-independent in each $M_i$ and each $|E_i| \geq 7$. Also, if $M_1$ and $M_2$ satisfy these conditions, then $M_1 \oplus_3 M_2$ is the modular sum operation from [2]. See [14, Proposition 3.4] for a verification of Proposition 2.10.

**Proposition 2.10.** If $\Sigma$ is a signed graph such that $M(\Sigma)$ is regular and $\Upsilon$ is a balanced signed graph, then $M(\Sigma \oplus_3 \Upsilon) = M(\Sigma) \oplus_3 M(\Upsilon)$ (or $M(\Sigma) \triangle M(\Upsilon)$ when the 3-sum is not defined).

**Proposition 2.11.** If $M_1$ is a regular signed-graphic matroid and $M_2$ is a graphic matroid, then $M_1 \oplus_3 M_2$ is signed-graphic.

**Proof.** It is known that the class of graphic matroids is closed under 3-summing, so assume that $M_1$ is not graphic. Now say that $M_1 = M(\Sigma)$, $M_2 = M(G)$, and $T$ is the 3-point line along which the 3-sum is taken. Since $M(\Sigma)$ is not graphic, $\Sigma$ is tangled (Theorem 2.4) and so $T$ is a positive triangle in $\Sigma$ because $\Sigma$ is loopless (Proposition 2.2). Now let $\Upsilon$ be the signed graph with underlying graph $G$ and all edges signed positively. Note that $T$ is a positive triangle in $G$ and so, by Proposition 2.10, $M_1 \oplus_3 M_2 = M(\Sigma) \oplus_3 M(\Upsilon) = M(\Sigma \oplus_3 \Upsilon)$, as required. \[ \square \]

**Proposition 2.12.** If $G_1$ is a vertically 2-connected graph, $G_2$ is either a vertically 2-connected graph or tangled signed graph, and both are as shown in Figure 2, then the parallel connection of $M^*(G_1)$ with $M^*(G_2)$ along the triad $T$ is $M^*(H)$, where $M^*(H)$ is the cographic matroid of $H$ when $H$ is a graph and is the dual of the signed-graphic matroid of $H$ when $H$ is a signed graph.

**Proof.** Consider $G_1$ to be an all-positive signed graph when $G_2$ is a signed graph. A circuit $C$ in $H$ is either a positive circle or a one-vertex join of two negative circles. In the former case, $C$ is either a positive circle in some $G_i \setminus T$ or $C = C_1 \cup C_2$, where $C_i$ is a positive circle in $G_i$ and $C_1 \cap C_2$ consists of two edges of $T$. In the latter case, $C$ is either the one-vertex join of two negative circles in $G_2 \setminus T$ or $C = C_1 \cup C_2 \cup D$, where $D$ is a negative circle in $G_2 \setminus T$, $C_1$ is a positive circle in $G_1$, $C_2$ is a negative circle in $G_2$, and $C_1 \cap C_2$ consists
of two edges of $T$. So if $C = C_1 \cup \ldots \cup C_n$ is a union of circuits in $M(H)$, then there is a corresponding union of circuits $C_i$ in $M(G_i)$. Thus $E(H) \setminus C = (E(G_1) \setminus C_1) \cup (E(G_2) \setminus C_2)$. Thus a flat in $M^*(H)$ is a flat in the parallel connection $P(M^*(G_1), M^*(H))$. Conversely, any flat in the parallel connection is a flat of one of the terms or the union of a flat from each term with a common intersection in $T$. Similarly, we can show that any flat in the parallel connection will correspond in the same way to a flat in $M^*(H)$. 

3. Excluded minors that are 2-connected but not 3-connected

A collection $\mathcal{N}$ of connected matroids is called 1-rounded when any connected matroid $M$ containing a minor from $\mathcal{N}$ satisfies the following: for every $e \in E(M)$, $M$ has a minor from $\mathcal{N}$ that uses $e$.

**Theorem 3.1 (Seymour [9]).** The collection $\{U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3}), M^*(K'_{3,3})\}$ is 1-rounded. Here $K'_{3,3}$ denotes the graph obtained from $K_{3,3}$ by adding a link joining two non-adjacent vertices.

**Theorem 3.2.** If $M$ is 2-connected, not 3-connected, and a regular excluded minor for the class of signed-graphic matroids, then $M \in \{M^*(G_1), \ldots, M^*(G_{29})\}$.

**Proof.** Since $M$ is 2-connected but not 3-connected, $M = M_1 \oplus M_2$. Let $e$ be the edge along which the 2-sum is taken. Since $M$ is minor minimal, each $M_i$ is signed-graphic and hence not graphic by Proposition 2.9. Now, since $M_i$ is regular it does not contain any of $U_{2,4}, F_7,$ and $F_7^*$ as a minor; however, being not graphic implies that $M_i$ contains either $M^*(K_5)$ or $M^*(K_{3,3})$ as a minor. Theorem 3.1 now implies that each $M_i$ contains an $M^*(H_i)$ minor where $H_i \in \{K_5, K_{3,3}, K'_{3,3}\}$ and $e \in H_i$. Thus $M$ contains $M^*(H_1) \oplus M^*(H_2) = M^*(H_1 \oplus H_2)$ as a minor and one can check that the 6 possibilities for $H_1 \oplus H_2$ are all in $\{G_1, \ldots, G_{29}\}$. So, since $M$ is minor minimal, $M = M^*(H_1 \oplus H_2) \in \{M^*(G_1), \ldots, M^*(G_{29})\}$. 

4. The matroids $R_{15}$ and $R_{16}$

Consider the graphs $G_1$ and $H_1$ in Figure 3. (Note that $G_1$ is one of the 29 vertically 2-connected excluded minors for the class of projective-planar graphs and, by Proposition 2.12, $M^*(H_1)$ is the parallel connection of $M^*(K_4)$ and $M^*(K_{3,3})$ along a triad in each graph.) Let $T$ be the 3-edge bond separating the two copies of $K_{2,3}$ in $G_1$. The matroid $R_{15}$
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Figure 4.

is obtained from $M^*(G_1)$ by a $\Delta Y$ exchange along $T$ (i.e., $R_{15} = M^*(G_1) \triangle M(K_4)$). Also, $R_{15} = M^*(K_{3,3}) \oplus_3 M^*(H_1)$, where the 3-sum is along a triad in $K_{3,3}$ and the 3-edge bond $T'$ in $H_1$ separating the triangle and the copy of $K_{2,3}$. $R_{15}$ has 15 elements and rank 7.

The matroid $R_{16}$ is obtained by taking two edge-disjoint triangles of $M(K_5)$ and 3-summing a copy of $M^*(K_{3,3})$ along each of the two triangles. The matroid $R_{16}$ has 16 elements and rank 8.

**Proposition 4.1.** $R_{15}$ is not signed-graphic.

**Proposition 4.2.** Any proper minor of $R_{15}$ is signed-graphic.

**Proposition 4.3.** $R_{16}$ is not signed-graphic.

**Proposition 4.4.** Any proper minor of $R_{16}$ is signed-graphic.

Propositions 4.1–4.4 show that $R_{15}$ and $R_{16}$ are both excluded minors for the class of signed-graphic matroids. In the remainder of this section we prove these four propositions.

**Proposition 4.5.** The matroids $R_{15}$ and $R_{16}$ are both 3-connected, not graphic, and not cographic.

**Proof.** $R_{15}$ and $R_{16}$ are both 3-connected as each is a 3-sum of two 3-connected matroids. Both $R_{15}$ and $R_{16}$ are not graphic because each contains an $M^*(K_{3,3})$ minor. $R_{16}$ is not cographic because it contains an $M(K_5)$ minor. Lastly, we show that $R_{15}$ is not cographic by displaying an $M(K_{3,3})$ minor. By deleting one and contracting two edges in the $K_{2,3}$ subgraph of $H_1$ we obtain the triangular prism $P$ (i.e., two vertex-disjoint triangles connected by three links) without disturbing the 3-edge bond $T'$ in $H_1$. Note that $M^*(P) = M(K_5 \setminus e)$, and so $R_{15} = M^*(K_{3,3}) \oplus_3 M^*(H_1)$ contains $M^*(K_{3,3}) \oplus_3 M(K_5 \setminus e)$ as a minor. Now $M^*(K_{3,3})$ contains an $M(K_4)$ minor using any of the triads of $K_{3,3}$. So now $R_{15}$ contains $M(K_4) \triangle M(K_5 \setminus e) = M(K_4 \oplus_3 K_5 \setminus e) = M(K_{3,3})$ as a minor.

The matroid $R_{12}$ is defined as $M^*(K_{3,3}) \oplus_3 M(K_5 \setminus e)$, where the 3-sum is along a triad in $K_{3,3}$ and the separating triangle in $K_5 \setminus e$. Let $\Sigma_{3,3}$ and $\Sigma_{12}$, respectively, be the signed graphs in Figure 4.
Proposition 4.6 (Zaslavsky [20, Proposition 4A]). If \( \Sigma \) is a signed graph without isolated vertices, then \( M(\Sigma) \cong M^*(K_{3,3}) \) if and only if \( \Sigma \cong \Sigma_{3,3} \).

Proposition 4.7. If \( \Sigma \) is a signed graph without isolated vertices, then \( M(\Sigma) \cong R_{12} \) if and only if \( \Sigma \cong \Sigma_{12} \).

In order to prove Proposition 4.7 we need Propositions 4.8–4.10. Proposition 4.8 is implied from the main result of [13] and Theorem 1.4.

Proposition 4.8 (Slilaty [13]). If \( \Sigma \) is a tangled signed graph without isolated vertices such that \( M(\Sigma) \) is 3-connected, not graphic, not cographic, and not \( R_{10} \), the \( \Sigma = \nabla \oplus_3 G \) where \( \nabla \) is tangled, \( G \) is all positive, and \( G \) has at least 5 vertices.

Proposition 4.9. If \( M_1 \) and \( M_2 \) are binary matroids and \( M_1 \oplus_3 M_2 \) is 3-connected and contains an \( M(K_5) \) minor, then either \( M_1 \) or \( M_2 \) contains an \( M(K_5) \) minor.

Proof. Suppose that \( N \cong M(K_5) \) is a minor of \( M_1 \oplus_3 M_2 \) such that each \( A_i = E(N) \cap E(M_i) \neq \emptyset \). Because \( (E(M_1), E(M_2)) \) is a 3-separation of \( M_1 \oplus_3 M_2 \) and this 3-separation induces a separation in any minor, \( r_\kappa(A_1) + r_\kappa(A_2) - r(N) \leq 2 \). Since \( N \) is internally 4-connected, \( |A_1| \leq 3 \) or \( |A_2| \leq 3 \), assume the former. Also, since \( N \) is simple and has no triads, \( |A_1| \leq 3 \) implies \( A_1 \) is a triangle of \( N \) or an independent set of one or two elements. Now, when viewing \( M_1 \) and \( M_2 \) as binary matrices where the triangle along which the 3-sum \( M_1 \oplus_3 M_2 \) is taken contains three non-zero rows, we see that \( M(K_5) \) is a minor of \( M_2 \). \( \square \)

Proposition 4.10. If \( \Sigma \) is an unbalanced signed graph without isolated vertices, then \( M(\Sigma) \cong M(K_5) \) if and only if \( \Sigma \) is one of the signed graphs in Figure 5.

Proof of Proposition 4.10. Since \( M(K_5) \) is 3-connected, it follows from Propositions 2.7 and 2.8 that \( \Sigma \) is vertically 2-connected. It then follows from [14, Theorem 2.6] that \( \Sigma \) either has a balancing vertex, is balanced after removing any negative loops, or is tangled. In the first case, it follows from Proposition 2.6 that \( \Sigma \) is the signed graph of Figure 5(a). In the second case, it follows from [14, Proposition 2.2] that \( \Sigma \) is the signed graph of Figure 5(b). If \( \Sigma \) is tangled, then by Proposition 2.2 \( \Sigma \) has no negative loops. So since
$M(K_5)$ is simple with 10 elements and rank four, $\Sigma$ has at least four negative digons on four vertices. This will force two vertex-disjoint negative digons, a contradiction. □

Proof of Proposition 4.7. Suppose that $R_{12} = M(\Sigma)$. Since $R_{12}$ is 3-connected and not graphic (because $R_{12}$ has an $M'(K_{3,3})$ minor), $\Sigma$ is tangled and vertically 3-connected (Theorems 2.4 and 2.5). Since $R_{12} = M'(K_{3,3}) \oplus_3 M(K_5 \setminus e)$ contains $M(K_4) \oplus_3 M(K_5 \setminus e) = M(K_4 \oplus_3 K_5 \setminus e) = M(K_{3,3})$ as a minor, $R_{12}$ is not cographic. Thus $\Sigma = \overline{Y} \oplus_3 G$ as in Proposition 4.8, with at least 5 vertices in $G$. It cannot be that $M(\overline{Y})$ is graphic, because otherwise $M(\Sigma) = M(\overline{Y} \oplus_3 G) = M(\overline{Y}) \oplus_3 M(G)$ would be graphic, a contradiction. Thus $r(\overline{Y}) \geq 4$ and so, since $|V(\Sigma)| = 6$ and $|V(G)| \geq 5$, we have that $|V(\overline{Y})| = 4$ and $|V(G)| = 5$.

Since $|V(\overline{Y})| = 4$, it must follow that $M(\overline{Y})$ is cographic unless $M(\overline{Y})$ contains $M(K_5)$ as a submatroid. But then Proposition 4.10 would imply that $\overline{Y}$ would contain one of the signed graphs of Figure 5 as a subgraph. But then $\Sigma = \overline{Y} \oplus_3 G$ would have a negative loop, a contradiction of tangledness.

Also, since $|V(\overline{Y})| = 4$ and $M(\overline{Y})$ is not graphic, it must be that $M(\overline{Y})$ contains $M'(K_{3,3})$ as a submatroid. Thus $M(\overline{Y}) = M'(H)$, where $H$ is a decontraction of $K_{3,3}$; that is, $H$ is a subdivision of $K_{3,3}$. Note that any 3-edge bond $T$ of $H$ contains three links from three incident branches of $H$. Using Whitney 2-isomorphisms we can then assume that $T$ is the set of links incident to a 3-valent vertex of $H$.

Now since $|V(G)| = 5$ it must be that $G$ is planar unless $G$ contains a $K_5$ subgraph. However, $G$ contains no more than 9 edges because $|E(\overline{Y})| \geq 9$ and $|E(\Sigma)| = 12$. Now let $a$, $b$, and $c$ be the vertices of $G$ along which the 3-sum with $\overline{Y}$ is taken, and let $x$ and $y$ be the remaining two vertices of $G$. Either $x$ and $y$ are adjacent or not. Let these be Cases 1 and 2, respectively.

Case 1. If $x$ and $y$ are adjacent, then since $|E(G)| \leq 9$ we can assume that the simplification of $G$ is contained in $K_5 \setminus e$ where without loss of generality $e$ is the $xa$-link. Now, if $T$ is the edge set of the triangle in $K_5 \setminus e$ on vertices $a, b, c$, then $K_5 \setminus e$ is planar and the planar dual graph $(K_5 \setminus e)^*$ has $T$ as a vertex bond. So now $R_{12}$ is contained in the parallel connection of $M'(H)$ and $M'(K_5 \setminus e)$ along a 3-valent vertex in each term. By Proposition 2.12, this parallel connection and then $R_{12}$ are both cographic, a contradiction.

Case 2. If $x$ and $y$ are not adjacent, then since $\Sigma$ is vertically 3-connected, each of $x$ and $y$ is adjacent to all of $a, b,$ and $c$. Thus $G$ simplifies to $K_5 \setminus e$ where $x$ and $y$ are the 3-valent vertices. So we have that $|E(G)| \geq 9$ and so, since $|E(\overline{Y})| \geq 9$, we get that $|E(G)| = |E(\overline{Y})| = 9$ and so $M(\overline{Y}) \cong M'(K_{3,3})$ and $G = K_5 \setminus e$. Thus $\overline{Y} \cong \Sigma_{3,3}$ by Proposition 4.6 and so $\Sigma = \overline{Y} \oplus_3 G \cong \Sigma_{12}$. □

Proposition 4.11. $R^*_1$ has exactly one triangle $A$ and $R^*_1/A \cong M(K_{3,4})$.

Proof. $R^*_1$ is obtained from a $Y \Delta$ switch of $M(G_1)$ along the 3-edge bond $T$. So now $R^*_1 = M(G_1) \Delta M(K_4)$. Let $P$ be the prism graph that consists of two vertex-disjoint triangles joined by three links. Thus $G_1 = (K_5 \setminus e) \oplus_3 P \oplus_3 (K_5 \setminus e)$, and so

$$R^*_1 = M(K_5 \setminus e) \oplus_3 (M(P) \Delta M(K_4)) \oplus_3 M(K_5 \setminus e).$$
Now $M(P) \triangle M(K_4) = M^*(K_{3,3})$ because

$$(M(P) \triangle M(K_4))^* = M^*(P) \triangle M^*(K_4) = M(K_5 \backslash e) \triangle M^*(K_4) = M(K_{3,3}),$$

and so

$$R_{15}^* = M(K_5 \backslash e) \oplus_3 M^*(K_{3,3}) \oplus_3 M(K_5 \backslash e).$$

Using signed-graphic 3-sums and the signed graph $\Sigma_{3,3}$ in Figure 4, we get that the signed graph in Figure 6(a), say $\Sigma$, has $M(\Sigma) \sim R_{15}^*$.

By inspection we see that there is only one triangle $A$ in $M(\Sigma) \sim R_{15}^*$. Now $R_{15}^*/A = M(\Sigma)/A = M(\Sigma/A)$, and $\Sigma/A$ is the signed graph in Figure 6(b). The graph $G_v$ from Proposition 2.6 obtained from $\Sigma/A$ is $K_{3,4}$, and so $M(\Sigma/A) = M(K_{3,4})$.

**Proof of Proposition 4.1.** Suppose that $R_{15} = M(\Sigma)$, where $\Sigma$ has no isolated vertices. Since $R_{15}$ is 3-connected and is neither graphic nor cographic (Proposition 4.5), $R_{15}$ contains an $R_{12}$-minor by [10, (7.4) and (14.2)]. Thus $\Sigma$ contains a $\Sigma_{12}$ minor by Proposition 4.7. So since $R_{15}$ and $R_{12}$ are both 3-connected, there are three edges $c,d,e$ in $\Sigma$ such that $\Sigma_{12} \cong \Sigma/c \backslash d \backslash e$.

Consider the labelling of the vertices of $\Sigma/c \backslash d \backslash e \cong \Sigma_{12}$ in Figure 7. Since $\Sigma$ must be vertically 3-connected (by Theorem 2.5), $\Sigma$ has minimum degree three.

**Claim 1.** $\Sigma$ has minimum degree four.

**Proof of Claim.** Since $\Sigma$ is vertically 3-connected and has no balancing vertex, the edges incident to any vertex $v$ form a cocircuit of $M(\Sigma)$. So Proposition 4.11 implies that $\Sigma$ has at most one 3-valent vertex. So, by way of contradiction say that $v$ is a 3-valent vertex of
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Figure 8.

$\Sigma$. Thus the 3-edge set $A$ in Proposition 4.11 is the collection of edges incident to $v$ and so $M^*(K_{3,4}) \cong R_{15} \setminus A = M(\Sigma \setminus v)$. Since $K_{3,4}$ is non-planar, it follows from [12, Theorem 3] that $\Sigma \setminus v$ is the projective-planar dual signed graph of some embedding of $K_{3,4}$ in the projective plane. If $G$ is embedded in the projective plane, then the projective-planar dual signed graph of $G$ is $(G^*, \sigma)$, where $G^*$ is the topological dual graph of $G$ and $\sigma$ is a signing such that a circle $C$ in $G^*$ is positive if and only if $C$ bounds a disk in the projective plane. Up to isomorphism of $K_{3,4}$, the only embedding $K_{3,4}$ in the projective plane is that shown in Figure 8, and thus $\Sigma \setminus v$ is isomorphic to the signed graph in Figure 8.

Note that $\Sigma \setminus v$ is vertically 4-connected. Now since $R_{15}$ is neither graphic, cographic, nor $R_{10}$, Proposition 4.8 implies that $\Sigma$ has a 3-separation $(A, B)$ in which $\Sigma : B$ is balanced with at least five vertices. Since $\Sigma \setminus v$ is vertically 4-connected, we must then have that all vertices of $\Sigma \setminus v$ are in $\Sigma : B$. Given this, it is easily seen that no such 3-separation exists, a contradiction.

Given Claim 1, the 3-valent vertices $x$ and $y$ in $\Sigma/c \setminus e$ must have degree at least four in $\Sigma/c$. Now since $M(\Sigma)$ is 3-connected, $M(\Sigma/c)$ is 2-connected, regular, and contains an $R_{12}$-minor. Thus $\Sigma/c$ is tangled and loopless. Thus $d$ and $e$ are both links in $\Sigma/c$ and, since $r(\Sigma/c) = 6$, $\Sigma/c$ has the six vertices as in Figure 7. Without loss of generality say that $d$ is incident to $x$ in $\Sigma/c$. It cannot be that $d$ is a positive link between $x$ and $y$, because otherwise $\Sigma/c$ would contain a $K_5$ minor, which would make $M(\Sigma) = R_{15} = M^*(H_1) \oplus_3 M^*(K_{3,3})$ contain an $M(K_5)$ minor and then Proposition 4.9 would imply that one of the cographic matroids $M^*(H_1)$ and $M^*(K_{3,3})$ contains an $M(K_5)$ minor, a contradiction. Furthermore, it cannot be that $d$ is a negative link between $x$ and $y$, because then $\Sigma/c$ contains two vertex-disjoint negative circles, a contradiction of tangledness. Also, it cannot be that $d$ is a link from $x$ to $z$ of either sign, because again we would have two vertex-disjoint negative circles. Finally, it cannot be that $d$ is a negative link from $x$ to a vertex in $\{l, m, n\}$, because we would again have two vertex-disjoint negative circles. Thus $d$ is a positive link from $x$ to a vertex in $\{l, m, n\}$. Similarly $e$ must be a positive link from $y$ to a vertex in $\{l, m, n\}$. Since $M(\Sigma)$ has no parallel elements, the end-points of $d$ and $e$ in $\{l, m, n\}$ are the same. Now, in decontracting $c$ to obtain $\Sigma$, we cannot leave parallel edges of the same sign and every vertex must have degree at least four (Claim 1), making the only possibilities for $\Sigma$ those given in Figure 9. (Using symmetries including switching at $z$, flipping horizontally, and permuting the end-points of $c$, the reader can verify that
these are indeed all of the possibilities for $\Sigma$.) In all of these signed graphs, there are two vertex-disjoint negative circles, a contradiction. □

Proof of Proposition 4.2. We show that $R_{15}/e$ and $R_{15}/e$ are both signed-graphic for any $e \in R_{15}$. By symmetry there are two cases to check: when $e$ is in one of the copies of $K_{2,3}$ of $G_1$, and when $e$ is obtained from the $\Delta Y$-exchange on $G_1$. In the former case, $G_1/e$ and $G_1/e$ are both projective-planar. Thus $M^*(G_1/e)$ and $M^*(G_1/e)$ are both signed-graphic by Theorem 1.4, and so $R_{15}/e = M^*(G_1/e)\triangle M(K_4)$ and $R_{15}/e = M^*(G_1/e)\triangle M(K_4)$ are both signed-graphic by Proposition 2.11. In the latter case, $R_{15}/e = M^*(G_1)\triangle M(K_4/e)$ is a 1-edge deletion of $M^*(G_1)$ and $R_{15}/e = M^*(G_1)\triangle M(K_4/e)$ is a subdivision of a 2-edge deletion of $M^*(G_1)$, and so both are signed-graphic by Theorem 1.4. □

Proposition 4.12. $R_{16}^*$ has no triangles.

Proof. Let $-W_5$ be the signed graph in Figure 10(a). In [20, Proposition 4A] it is shown that $M(-W_5) = M^*(K_5)$. Since $R_{16} = M^*(K_{3,3}) \oplus_3 M(K_5) \oplus_3 M^*(K_{3,3})$, we can switch edges as necessary and use Proposition 2.12 to get that the signed graph in Figure 10(b), say $\Upsilon$, satisfies $M(\Upsilon) = R_{16}^*$. Evidently $M(\Upsilon)$ has no triangles. □
Define the matroid $R_{13} = M^*(K_{3,3}) \oplus_3 M(K_5)$ and the signed graph $\Sigma_{13}$ to be $\Sigma_{12} \cup e$ in which $\Sigma_{12}$ is as shown in Figure 7 and $e$ is a positive $xy$-link. Proposition 4.13 can be deduced from Propositions 4.7 and 4.9.

**Proposition 4.13.** If $\Sigma$ is a signed graph without isolated vertices, then $M(\Sigma) \cong R_{13}$ if and only if $\Sigma \cong \Sigma_{13}$.

**Proof of Proposition 4.3.** Say that $R_{16} = M(\Sigma)$, where $\Sigma$ has no isolated vertices. Since $\Sigma$ must be vertically 3-connected (Theorem 2.5) and has no balancing vertex (Theorem 2.4), the set of edges incident to any vertex is a cocircuit of $M(\Sigma)$. So, since $R_{16}$ has no triangles (Proposition 4.12), $\Sigma$ has minimum degree at least four. Since the number of edges in $\Sigma$ is 16, $\Sigma$ must then be 4-regular.

Since $R_{16} = M^*(K_{3,3}) \oplus_3 M(K_5) \oplus_3 M^*(K_{3,3})$ has an $R_{13} = M^*(K_{3,3}) \oplus_3 M(K_5)$ minor, $\Sigma$ must have a $\Sigma_{13}$ minor by Proposition 4.13. Thus there are edges $c_1,c_2,d$ such that $\Sigma_{13} \cong \Sigma/c_1/c_2/d$. Since $\Sigma$ is 4-regular, the degree sequence of $\Sigma/c_1$ is 4/4/4/4/4/4/6, and so the degree sequence for $\Sigma/c_1/c_2$ is either 4/4/4/4/4/6/6 or 4/4/4/4/4/8. Let these be Cases 1 and 2, respectively. In each case use the labelling of $\Sigma/c_1/c_2/d \cong \Sigma_{13}$ in Figure 7. Note that the degree sequence of $\Sigma/c_1/c_2/d$ is 4/4/4/4/4/6.

**Case 1.** Here $d$ must be a loop on one of the 6-valent vertices of $\Sigma/c_1/c_2$, say vertex $v$, and that loop must be positive by Proposition 2.2. Without loss of generality, decontracting $c_1$ splits $v$ and decontracting $c_2$ splits the other 6-valent vertex of $\Sigma/c_1/c_2$. However, then $c_1$ and $d$ will be parallel positive links in $\Sigma$, a contradiction of the 3-connectedness of $M(\Sigma)$.

**Case 2.** Here $d$ is a loop on the 8-valent vertex of $\Sigma/c_1/c_2$, say $v$, and by Proposition 2.2, $d$ is positive. So now decontracting $c_1$ and $c_2$ each splits $v$, and so the only possibilities for $\Sigma$ are as shown in Figure 11.

All of these signed graphs except Figure 11(m) have two vertex-disjoint negative circles. The vertex-disjoint negative circles are easily seen in all of cases except perhaps the last two, where we have marked one negative quadrilateral using $x$ and the other negative quadrilateral is unmarked. Thus $\Sigma$ is the signed graph of Figure 11(m). Note that $\Sigma = Y \oplus_3 K_5$, where $Y$ is the signed graph of Figure 8 satisfying $M(Y) = M^*(K_{3,4})$. Thus $R_{16} \cong M(\Sigma) \cong M^*(K_{3,4}) \oplus_3 M(K_5)$; however, $R_{16} = M^*(K_{3,3}) \oplus_3 M(K_5) \oplus_3 M^*(K_{3,3})$, as defined. As in the proof of Proposition 4.12, the signed graph of Figure 10(b), say $Y_1$, satisfies $M^*(Y_1) \cong M^*(K_{3,3}) \oplus_3 M(K_5) \oplus_3 M^*(K_{3,3})$. In a similar fashion, the signed graph in Figure 10(c), say $Y_2$, satisfies $M^*(Y_2) \cong M^*(K_{3,4}) \oplus_3 M(K_5)$. Thus $M(Y_1) \cong M(Y_2)$.

**Claim 1.** If the negative edges of a signed graph $\Omega$ form a triangle, then $M(\Omega)$ is graphic.

**Proof of Claim.** Let $G$ be the ordinary graph obtained from $\Omega$ by replacing the all-negative triangle with a triad on the same vertices. It is easy to check that $M(G) \cong M(\Omega)$. 

**Claim 2.** $Y_1$ has exactly four vertex-deleted subgraphs whose matroids are not graphic and $Y_2$ has exactly three vertex-deleted subgraphs whose matroids are not graphic.
Proof of Claim. For $\mathcal{Y}_1$, if $v$ is one of its four 3-valent vertices, then $M(\mathcal{Y}_1 \setminus v)$ is not graphic because it contains a $M(-W_5) = M^*(K_5)$ submatroid. If $w$ is any other vertex of $\mathcal{Y}_1$, then either $\mathcal{Y}_1 \setminus w$ has a balancing vertex or the negative edges of $\mathcal{Y}_1 \setminus w$ form a triangle. In either case $M(\mathcal{Y}_1 \setminus w)$ is graphic by Claim 1 and Proposition 2.6.

For $\mathcal{Y}_2$, if $v$ is one of the three 3-valent vertices in the upper left, then $M(\mathcal{Y}_2 \setminus v)$ is not graphic because it contains a $M(-W_5) = M^*(K_5)$ submatroid. If $w$ is any other vertex of $\mathcal{Y}_2$, then $M(\mathcal{Y}_2 \setminus w)$ is graphic because either $\mathcal{Y}_2 \setminus w$ has a balancing vertex or the negative edges of $\mathcal{Y}_2 \setminus w$ form a triangle after switching.

Given the form of the rank function of a signed-graphic matroid $M(\mathcal{Y})$, a cocircuit is a disjoint union $S \cup B$, where $S$ is a set of edges separating $\mathcal{Y}:X$ from $\mathcal{Y}:Y$ and $B$ is a minimal balancing set of $\mathcal{Y}:X$. Thus a connected hyperplane of $M(\mathcal{Y})$ is the complement of $S \cup B$ only if either $S = \emptyset$ or $\mathcal{Y}:X$ has only one vertex. That is when $S \cup B$ is a minimal
balancing set of \( \Upsilon \), or \( S \cup B \) is the collection of edges incident to a single vertex and that vertex is not a balancing vertex. So by Claim 2, \( M(\Upsilon_1) \) has exactly four connected non-graphic hyperplanes and \( M(\Upsilon_2) \) has exactly three connected non-graphic hyperplanes. Thus \( M(\Upsilon_1) \not\cong M(\Upsilon_2) \), a contradiction. \( \square \)

**Proof of Proposition 4.4.** We show that \( R_{16}\setminus e \) and \( R_{16}/e \) are both signed-graphic for any \( e \in R_{16} \). By symmetry there are two cases to check: \( e \) is an element of one of the two \( M^*(K_{3,3}) \) terms and \( e \) is an element of the \( M(K_5) \) term. In the former case \( M^*(K_{3,3}\setminus e) \) and \( M^*(K_{3,3}/e) \) are both graphic. So since \( M(K_5) \oplus_3 M^*(K_{3,3}) \cong M(\Sigma_{13}) \), we get that \( R_{16}\setminus e = M^*(K_{3,3}/e) \oplus_3 M(\Sigma_{13}) \) and \( R_{16}/e = M^*(K_{3,3}\setminus e) \oplus_3 M(\Sigma_{13}) \) are both signed-graphic by Proposition 2.11.

In the latter case, \( K_5\setminus e \) is planar and neither of the two triangles along which the 3-sums are taken is the separating triangle of \( K_5\setminus e \). Thus \( M^*(K_{3,3}) \oplus_3 M(K_5\setminus e) = M^*(K_{3,3}) \oplus_3 M^*(P) \), where \( P = (K_5\setminus e)^* \) is the triangular prism and where the 3-sum is along a triad in each term. Thus \( M^*(K_{3,3}) \oplus_3 M^*(P) \) is the cographic matroid as given in Proposition 2.12. So now, by Proposition 2.12, \( R_{16}\setminus e = M^*(K_{3,3}) \oplus_3 M^*(P) \oplus_3 M^*(K_{3,3}) \) is the cographic matroid of the graph in Figure 12(a). One can check that this graph is projective-planar and so \( R_{16}\setminus e \) is signed-graphic by Theorem 1.4. By a similar argument \( R_{16}/e \) is the cographic matroid of the graph in Figure 12(b) and that graph is projective-planar. Thus \( R_{16}/e \) is signed-graphic by Theorem 1.4. \( \square \)

### 5. Lemmas for Section 6

#### 5.1. Lemmas on graphs with \( K_{3,3} \) minors

**Lemma 5.1 (Truemper [16, 10.3.9]).** Let \( G \) be a graph containing a \( K_{3,3} \) minor such that \( M(G) \) is 3-connected.

1. If \( G \) contains a triangle with edge set \( \{e_1,e_2,e_3\} \), then \( G \) has one of the graphs in Figure 13 as a minor, where \( \{e_1,e_2,e_3\} \) is shown in bold.
2. If \( G \) has a vertex \( v \) of degree 3, then \( G \) contains a subdivision of \( K_{3,3} \) that uses \( v \) as one of its branch vertices.

**Lemma 5.2.** Let \( G \) be a graph with a \( K_{3,3} \) minor such that \( M(G) \) is 3-connected. If \( G \) contains a 3-edge matching \( T \) that is also a bond of \( G \), then \( G \) has an \( H_1 \) minor (see Figure 3) in which \( T \) is the 3-edge bond in \( H_1 \) that separates the triangle and the copy of \( K_{2,3} \).
Figure 13.

**Proof.** Since \( M(G) \) is 3-connected, we can partition \( E(G) \) into \( A, T, B \), where \( A \) and \( B \) are the edge sets of two components of \( G \setminus T \). Now if \( H \) is a subdivision of \( K_{3,3} \) in \( G \), then since \( H \) is vertically 2-connected \( H \) either uses all the edges of \( T \), two edges of \( T \), or no edges of \( T \). In the first case, without loss of generality, one branch vertex of \( H \) is contained in \( G \) and the rest of the branch vertices of \( H \) are contained in \( G \). In the second case, without loss of generality, \( H \) is either uses all the edges of \( T \), two edges of \( T \), or no edges of \( T \). In all three cases, \( G/A \) contains \( H/A \) and \( H/A \) is a subdivision of \( K_{3,3} \). Also, one may show that \( M(G/A) \) is 3-connected. Now let \( v \) be the vertex of \( G/A \) obtained by coalescing the vertices of \( A \). Evidently \( v \) has degree 3 in \( G/A \) and the edges incident to \( v \) are the edges of \( T \). Now, by Lemma 5.1, there is a subdivision \( \tilde{H} \) of \( K_{3,3} \) in \( G/A \) that has \( v \) as one of its 3-valent vertices. We can now split the vertex \( v \) to obtain an \( H_1 \) minor containing \( T \) as a 3-edge matching. 

**Lemma 5.3.** Let \( G \) be a graph such that \( M(G) \) is 3-connected and \( G \) contains a \( K_{3,3} \) minor, a triangle \( T \) on edges \( \{e_1, e_2, e_3\} \), and a 3-valent vertex \( v \) not in \( T \). Then \( G \) has one of the following:

1. a vertical 3-separation \( (A, B) \) with \( v \in V(A) \setminus V(B) \), \( V(T) \subseteq V(A) \), and \( |V(B)| \geq 5 \), or
2. one of the graphs in Figure 13 as a minor, where \( \{e_1, e_2, e_3\} \) is shown in bold and \( v \) is a 3-valent vertex not incident to \( \{e_1, e_2, e_3\} \).

**Proof.** By Lemma 5.1, there is a subdivision \( H \) of \( K_{3,3} \) in \( G \) that has \( v \) as a branch vertex. By Proposition 2.1 we can again choose \( H \) so that it has no local bridges and still contains \( v \) as a branch vertex. In this proof we will use the drawing and labelling of \( H \) shown in Figure 14. All edges in Figure 14 represent paths in \( H \) except those edges incident to \( v \) which are actual edges in \( H \). The cross-hatched edges represent paths in \( H \) that may have length zero. We use the terms ‘above’ and ‘below’ with respect to Figure 14.

Now, either all vertices of \( T \) are in \( H \) or all vertices of \( T \) are in the same \( H \)-bridge, say \( B_T \). If all vertices of \( T \) are in \( H \), then say that \( B_T = \emptyset \). Let \( A_T \) be the collection of attachments of \( B_T \), or if \( B_T = \emptyset \) then let \( A_T = V(T) \).

**Claim 1.** If \( A_T \) has a vertex above \( \{a, b, c\} \), then \( H \cup B_T \cup T \) contains a minor that satisfies part (2).

**Proof of Claim.** Since \( M(G) \) is 3-connected and \( A_T \) has a vertex above \( \{a, b, c\} \), we can choose a 3-element subset \( A \) of \( A_T \) such that: \( A \) contains \( V(T) \cap V(H) \), \( A \) contains
3 − |V(T) ∩ V(H)| other vertices chosen from among the attachments of BT, and A has at least one vertex above \{a, b, c\}. Note that if \( A = A_T = V(T) \), then not all vertices of T are on the same branch of H, because otherwise there is an edge of T that is a local 2-bridge of H, a contradiction. Also, since H has no local bridges, when \(|V(T) ∩ V(H)| < 3\) we can choose A so that not all of its vertices are on the same branch of H.

Since \( M(G) \) is 3-connected, we can use Menger’s theorem to obtain disjoint paths \( γ_1, γ_2, γ_3 \) in \( BT \) connecting \( V(T) \) to A. Now let \( \hat{H} \) be the graph obtained from \( H \cup T \cup γ_1 \cup γ_2 \cup γ_3 \) by contracting the edges of \( γ_1 \cup γ_2 \cup γ_3 \). That is, \( \hat{H} \) is obtained from \( H \) by placing the triangle T on the vertices in A. Now either there are two vertices of A on the same cross-hatched path in H or not. If not, then there exists \( C \subseteq E(H) \setminus E(T) \) such that \( \hat{H} / C \) is a subdivision \( H' \) of \( K_{3,3} \) along with the triangle \( T' = (\hat{H} / C) \cdot \{e_1, e_2, e_3\} \), where all three vertices of \( T' \) are branch vertices of \( H' \) other than v. We now have a minor, satisfying part (2) in \( \hat{H} / C \). In the former case, since the third vertex of A must then lie above \{a, b, c\}, there exist \( C \) and \( D \subseteq E(H) \setminus E(T) \) such that \( \hat{H} / C \setminus D \) is the graph in Figure 15, where \( e_1, e_2, e_3 \) are shown in bold.

In \( \hat{H} / C \setminus D \), if we contract the edge \( e' \), then we obtain a minor isomorphic to the graph in Figure 13(b), which satisfies part (2) of our conclusion.

Now if \( A_T \) has a vertex above \{a, b, c\}, then we are done by Claim 1. So suppose that all vertices of \( A_T \) are at and below \{a, b, c\}. Now rechoose H such that the total of the
lengths of the cross-hatched paths is a minimum and all vertices of $AT$ are at and below \( \{a, b, c\} \).

Let $H_0 = H \cup B_T \cup T$ and note that an $H_0$-bridge in $G$ is just an $H$-bridge that is not $B_T$ and not an edge of $T$. Let $B$ be the collection of $H_0$-bridges with attachments above $\{a, b, c\}$. If $B = \emptyset$, then we have a 3-separation of $G$ at $\{a, b, c\}$ satisfying part (1). Otherwise let $V$ be the subgraph of $H$ consisting of $v$ along with the three paths from $v$ to $\{a, b, c\}$. Let $d' = a$, or if there is a bridge in $B$ with an attachment on the $va$-path of $V$ below $a$, then let $d'$ be the lowest such attachment. Since $v$ has degree 3 in $G$, $d' \neq v$. Define $b'$ and $c'$ similarly. If $\{d', b', c'\} = \{a, b, c\}$, then again there is a 3-separation of $G$ at $\{a, b, c\}$ satisfying part (1). If not, then we get that $V'$ is a proper subgraph of $V$, where $V'$ is the subgraph of $V$ consisting $v$ along with the three paths from $v$ to $\{d', b', c'\}$. We will now show that there is a subdivision $H'$ of $K_{3,3}$ that contains $V'$ and $V$, and whose branch vertices include $v, a, b, c'$. After we have obtained $H'$, given that $V'$ is a proper subgraph of $V$ and $AT \subseteq V \subset H'$, we will get that the sum of the lengths of the cross-hatched paths of $H'$ is less than that sum for $H$. Hence there is a vertex of $AT$ above $\{d', b', c'\}$ in $H'$ and so we are done by Claim 1.

Now either there is an $H_0$-bridge in $B$ that contains $\{d', b', c'\}$, there is an $H_0$-bridge in $B$ that contains two elements from $\{d', b', c'\}$ and no bridge contains all three, or all $H_0$-bridges in $B$ contain at most one vertex from $\{d', b', c'\}$. In each case let $z$ be the $ad'$-path on $V$, let $\beta$ be the $bb'$-path on $V$, let $\chi$ be the $cc'$-path on $V$, let $Y$ be the subgraph of $H$ consisting of $y$ along with the three paths from $y$ to $\{a, b, c\}$, and let $Z$ be the subgraph of $H$ consisting of $z$ along with the three paths from $z$ to $\{a, b, c\}$. We will call a subdivision of $K_{1,3}$ a subdivided triad.

In the first case, let $Y'$ be a subdivided triad in this bridge that has $d', b', c'$ as leaf vertices (which must exist because $M(G)$ is 3-connected) and let $Z' = Z \cup z \cup \beta \cup \chi$, and we have that $H' = V' \cup Y' \cup Z'$ is a subdivision of $K_{3,3}$ that contains $V'$ and $V$ and whose branch vertices include $v, a, b', c'$. In the second case, without loss of generality, say that $B \in B$ contains $d'$ and $b'$, and that either $c' = c$ or $c' \neq c$ and there exists $C \in B$ that contains $c'$. By the definition of $B$, $B$ has an attachment on $H$ above $\{a, b, c\}$ and similarly for $C$. Assume without loss of generality that the attachment for $B$ is on $Y$. Now if $c' = c$ or the attachment for $C$ is also on $Y$, then there is a subdivided triad $Y''$ in $Y \cup B \cup C$ which has its 3-valent vertex in the interior of $B$, has leaf vertices $a, b', c'$, and intersects $z \cup \beta \cup \chi$ at $\{d', b', c'\}$ only. So now let $Z' = Z \cup z \cup \beta \cup \chi$, and we have that $H' = V' \cup Y'' \cup Z'$ is a subdivision of $K_{3,3}$ that contains $V'$ and $V$ and whose branch vertices include $v, a, b', c'$. If $c \neq c'$ and the attachment for $C$ is on $Z$, then there is a subdivided triad $Y'$ in $Y \cup B \cup C$ which has its 3-valent vertex in the interior of $B$, has leaf vertices $a, b', c'$, contains all of $\chi$, and intersects $z \cup \beta$ at $\{a, b'\}$ only. Also there is a subdivided triad $Z'$ in $Z \cup C \cup z \cup \beta$ which has 3-valent vertex in $C \cup Z$, whose leaf vertices are $a, b', c'$, contains $z \cup \beta$, and intersects $\chi$ at $c'$ only. We again have that $H' = V' \cup Y' \cup Z'$ is a subdivision of $K_{3,3}$ that contains $V'$ and $V$ and whose branch vertices include $v, a, b', c'$.

In the third case, either there is a bridge $B_a \in B$ that contains $d'$, or $d' = a$. We have a similar property for each of $b'$ and $c'$. Our desired conclusion follows in much the same fashion as in the previous paragraph.\qed
5.2. Other lemmas

Theorem 5.4 (Hall [5]). If $M(G)$ is 3-connected and $G$ contains a $K_5$ minor, then either $G \cong K_5$ (possibly along with some isolated vertices) or $G$ contains a $K_{3,3}$-subdivision.

It is almost true that each $M_i$ is 3-connected when $M_1 \oplus_3 M_2$ is 3-connected, the sole exception being some parallel elements along the triangle of summation (Proposition 5.5). Thus we can say that each $s_i(M_i)$ (i.e., the simplification of $M_i$) is 3-connected when $M_1 \oplus_3 M_2$ is 3-connected.

Proposition 5.5 (Seymour [10, (4.3)]). If $M_1 \oplus_3 M_2$ is 3-connected and $T$ is the triangle along which the 3-sum is taken, then each $M_i$ is 3-connected save perhaps for some elements parallel to elements of $T$.

Lemma 5.6. If $M_1 \oplus_3 M_2$ is 3-connected and each $M_i$ is cographic and not graphic, then either $M_1 \oplus_3 M_2$ is cographic or $M_1 \oplus_3 M_2$ has an $R_{15}$-minor.

Proof. Let $M_i = M^*(G_i)$ and say $T$ is the triangle along which the 3-sum is taken. Since $s_i(M_i)$ is 3-connected we can say that $G_i$ is a subdivision of a vertically 3-connected simple graph. Let $\hat{G}_i$ be obtained from $G_i$ by suppressing vertices of degree 2. Thus any 3-edge bond in $\hat{G}_i$ is either a vertex bond or a matching. Any series pair of edges in $G_i$ contains at most one edge from $T$, and so in suppressing vertices of degree 2 we need not contract any elements of $T$, and $T$ will still be a 3-edge bond of $\hat{G}_i$. Thus $\hat{G}_i:T$ is either a vertex bond or a matching. In the case that $\hat{G}_i:T$ is a vertex bond, $G_i:T$ is a vertex bond after some possible switching of series pairs of edges, and in the case that $\hat{G}_i:T$ is a matching, $G_i:T$ is a matching.

Since $M^*(\hat{G}_i)$ is 3-connected and not graphic, Theorem 5.4 implies that $\hat{G}_i \cong K_5$ or $G_i$ has a $K_{3,3}$-minor. Since $M^*(\hat{G}_i)$ has a triangle $T$, we must have the $K_{3,3}$-minor. Now if $G_i:T$ is a vertex bond, then Lemma 5.1 yields a $K_{3,3}$-minor of $G_i$ with $T$ as a vertex bond, and if $G_i:T$ is a matching, then Lemma 5.2 yields a $H_1$-minor of $G_i$ with $T$ as the matching bond. Now if $G_1:T$ and $G_2:T$ are both vertex bonds, then $M_1 \oplus_3 M_2$ is cographic by Proposition 2.12, and if, say, $G_1:T$ is a matching, then $M_1 \oplus_3 M_2$ contains $M^*(H_1) \oplus_3 M^*(K_{3,3}) = R_{15}$ as a minor, as required.

Lemma 5.7. If $M_1 \oplus_3 M_2$ is 3-connected and signed-graphic, and each $M_i$ is cographic and not graphic, then $M_1 \oplus_3 M_2$ is cographic.

Proof. Since $M_1 \oplus_3 M_2$ is signed-graphic it cannot contain an $R_{15}$-minor by Proposition 4.1. So now Lemma 5.6 implies that $M_1 \oplus_3 M_2$ is cographic.

Lemma 5.8. If $M$ is a 3-connected regular matroid of rank at least 3 that contains a triangle $T$, then there is a $M(K_4)$-minor of $M$ that contains $T$.

Proof. Bixby’s theorem (see [16, 3.4.36]) says that, for every element $e$ of $M$, either the simplification of $M/e$ or the cosimplification of $M\setminus e$ is 3-connected. One can obtain the
desired minor by continually applying this fact to elements outside the closure of $T$ until we reach rank 3. Once rank 3 is reached we have an $M(K_4)$-minor. □

The graph $2K_3$ is $K_3$ with each edge doubled. Lemma 5.9 follows immediately from Menger’s theorem.

**Lemma 5.9.** If $M(G)$ is 3-connected save possibly for some parallel edges, and $T_1$ and $T_2$ are two triangles in $G$, then there is an $M(2K_3)$-minor of $M(G)$ containing $T_1 \cup T_2$.

Given a 3-sum $M_1 \oplus_3 (M_2 \oplus_3 M_3)$ we say that $M_1$ sums into $M_2$ when the triangle of $M_2 \oplus_3 M_3$ along which the sum with $M_1$ is taken lies in $M_2$. Given $M_1 \oplus_3 (M_2 \oplus_3 M_3)$, if the triangle $\{e,f,g\}$ of $M_2 \oplus_3 M_3$ along which the 3-sum with $M_1$ is taken lies neither in $M_2$ nor $M_3$, then without loss of generality $M_2 \cap \{e,f,g\} = \{f,g\}$ and $M_3 \cap \{e,f,g\} = \{e\}$. In this case, however, as long as $|M_3| \geq 8$, we get $(M_2 \cup e) \oplus_3 (M_3 \setminus e) = M_2 \oplus_3 M_3$, and so $M_1 \oplus_3 (M_2 \oplus_3 M_3) = M_1 \oplus_3 (M'_2 \oplus_3 M'_3)$ and $M_1$ sums into $M'_2$. In the case that $M_1$ sums into $M_2$, we get that $M_1 \oplus_3 (M_2 \oplus_3 M_3) = (M_1 \oplus_3 M_2) \oplus_3 M_3$, where on the right side of the equation $M_3$ sums into $M_2$. When we write $M_1 \oplus_3 M_2 \oplus_3 M_3$ we mean $M_1 \oplus_3 (M_2 \oplus_3 M_3)$, where $M_1$ sums into $M_2$, and also $(M_1 \oplus_3 M_2) \oplus_3 M_3$, where $M_3$ sums into $M_2$. If the triangles $T_1$ and $T_3$ of $M_2$ along which the sums with $M_1$ and $M_3$ are taken satisfy $r_{M_2}(T_1 \cup T_3) = 2$, then we say that the three terms are summed along a common line.

**Lemma 5.10.** If $M_1 \oplus_3 M_2 \oplus_3 M_3$ is 3-connected, $M_1$ and $M_3$ are cographic and not graphic, $r(M_2) > 2$, and the three terms are summed along a common line, then $M_1 \oplus_3 M_2 \oplus_3 M_3$ contains an $R_{15}$-minor.

**Proof.** Let $T_1$ and $T_3$ be the triangles along which the sums $M_1 \oplus_3 M_2$ and $M_2 \oplus_3 M_3$ are taken. As in the proof of Lemma 5.6, for $i \in \{1, 3\}$, $M_i$ has a $M^*(K_{3,3})$-minor that contains $T_i$. By Lemma 5.8 we can find a $K$-minor of $M_2$ containing $T_1 \cup T_3$ where $K$ is $M^*(K_4)$ with the three edges of one triad subdivided and the resulting 6 edges are $T_1 \cup T_3$. So now $M_1 \oplus_3 M_2 \oplus_3 M_3$ has $M^*(K_{3,3}) \oplus_3 K \oplus_3 M^*(K_{3,3})$ as a minor, which by Proposition 2.12 is $M^*(H_1) \oplus_3 M^*(K_{3,3}) = R_{15}$.

**Lemma 5.11.** If $M = M_1 \oplus_3 M_2 \oplus_3 M_3$ is 3-connected, $M_1$ and $M_3$ are cographic and not graphic, and $M_2$ is graphic, then either:

1. $M_1 \oplus_3 M_2 \oplus_3 M_3$ is cographic,
2. $M_2 = N_1 \oplus_3 N_2$ where $M_1$ and $M_3$ sum into $N_1$ and $N_2$ is graphic and of rank at least 4, or
3. $M$ contains an $R_{15}$ or $R_{16}$-minor.

**Proof.** Let $T_1$ and $T_3$ be the triangles along which the sums $M_1 \oplus_3 M_2$ and $M_2 \oplus_3 M_3$ are taken. Let $M_i = M^*(G_i)$ for $i \in \{1, 3\}$ and let $M_2 = M(G_2)$. By Proposition 5.5, $M(G_2)$ is 3-connected except perhaps for some doubled edges with $T_1 \cup T_3$. In Case 1 say that $G_2$ is planar, and $T_1$ and $T_3$ are both facial triangles up to switching of parallel edges. In
Case 2 say that $G_2$ is non-planar. In Case 3 say that $G_2$ is planar and, without loss of generality, $T_1$ is a separating triangle of $G_2$.

**Case 1.** If $T_1$ and $T_3$ are facial triangles up to switching of parallel edge pairs, then $T_1$ and $T_3$ are vertex bonds in the planar dual graph $G_2^*$. As in the proof of Lemma 5.6, for each $i \in \{1, 3\}$ either $G_i:T_i$ is a vertex bond or a matching. In Case 1.1 say that both $G_1:T_1$ and $G_3:T_3$ are vertex bonds, in Case 1.2 say that $G_1:T_1$ is a vertex bond and $G_3:T_3$ is a matching, and in Case 1.3 both $G_1:T_1$ and $G_3:T_3$ are matchings.

**Case 1.1.** Here $M_1 \oplus_3 M_2 \oplus_3 M_3 = M^*(G_1) \oplus_3 M^*(G_2^*) \oplus_3 M^*(G_3)$ is cographic by Lemma 2.12.

**Case 1.2.** Here $M_3$ has an $M^*(H_1)$-minor as in Lemma 5.2 such that $H_1:T_3$ is a matching and $M_1$ has a $M^*(K_{3,3})$-minor that contains $T_1$ as a vertex bond. Using Lemma 5.9 on $T_1$ and $T_3$ in $M_2 = M(G_2)$, we then obtain $M^*(H_1) \oplus_3 M^*(K_{3,3}) = R_{15}$ as a minor of $M_1 \oplus_3 M_2 \oplus_3 M_3$.

**Case 1.3.** Here, for each $i \in \{1, 3\}$, $M_i$ has an $M^*(H_1)$-minor as in Lemma 5.2 such that $H_1:T_i$ is a matching. Thus $M_1$ has a $M^*(K_{3,3})$-minor in which $H_1:T_1$ is a vertex bond and so $M_1 \oplus_3 M_2 \oplus_3 M_3$ has an $R_{15}$-minor as in Case 1.2.

**Case 2.** By Theorem 5.4, $G_2$ is $K_5$ perhaps with some edges doubled or $G$ has a $K_{3,3}$-subdivision. Let these be Cases 2.1 and 2.2, respectively.

**Case 2.1.** Let $T_1$ and $T_3$ be the two triangles in $K_5$ along which the 3-sums with $M_1$ and $M_3$ are taken. If $|V(T_1) \cap V(T_3)| = 1$, then $M^*(K_{3,3}) \oplus_3 M(K_5) \oplus_3 M^*(K_{3,3}) = R_{16}$ is a minor of $M$. If $|V(T_1) \cap V(T_3)| = 3$, then $M_1 \oplus_3 M_2 \oplus_3 M_3$ are summed along a common line, and so by Lemma 5.10 $M_1 \oplus_3 M_2 \oplus_3 M_3$ has an $R_{15}$-minor. If $|V(T_1) \cap V(T_3)| = 2$, let $e$ be the link of $G_2$ that connects the two vertices of $T_1 \cup T_3$ not in $T_1 \cap T_3$. Now $T_1$ and $T_3$ share the same vertex set in $G_2/e$ and so $M_1 \oplus_3 (M_2/e) \oplus_3 M_3 = (M_1 \oplus_3 M_2 \oplus_3 M_3)/e$ has its three terms summed along a common line, and so by Lemma 5.10, $M_1 \oplus_3 M_2 \oplus_3 M_3$ has an $R_{15}$-minor.

**Case 2.2.** Let $G'$ be the graph obtained from $G_2$ by performing a $\Delta Y$ switch on $T_1$. If $v$ is the new 3-valent vertex in $G'$, then $v$ is not on the triangle $T_1$, and note also that $G'$ still has a $K_{3,3}$ subdivision. Now if part (1) of Lemma 5.3 holds for $G'$, then there is a 3-separation $(A, B)$ of $G$ with $T_1$ and $T_3$ in $A$ and $|V(B)| \geq 5$. Thus part (2) of the current lemma holds. If part (2) of Lemma 5.3 holds for $G'$, then one can check that $G_2$ has a $\hat{K}$ minor that contains $T_1 \cup T_3$, where either $\hat{K}$ is $K_5$ with some edges doubled, or $\hat{K}$ is $K_4$ with the edges of one triangle doubled and this doubled triangle is $T_1 \cup T_3$. As in Case 1, we will get that $M$ has either $R_{15}$ or $R_{16}$ as a minor.

**Case 3.** Let $G'$ be the graph obtained from $G_2$ by performing a $\Delta Y$ switch on $T_1$. Since $T_1$ is a separating triangle in $G_2$ and $G_2$ is planar, we now have that $G'$ contains a $K_{3,3}$-subdivision with the new 3-valent vertex, say $v$, as one branch vertex. In a similar fashion to Case 2.2, we will get either a 3-separation satisfying part (2) or an $R_{15}$ or $R_{16}$-minor. □
6. The 3-connected case

Let $M$ be an excluded minor for the class of signed-graphic matroids that is regular and 3-connected. Furthermore, by Theorem 1.3 assume that $M$ is not cographic. Now, by Theorem 1.2 and Proposition 5.5, $M = M_1 \oplus M_2$, where each $M_i$ is regular and signed-graphic and each $si(M_i)$ is 3-connected. By Proposition 2.11 and the minimality of $M$, neither $M_1$ nor $M_2$ is graphic. If we assume that both $M_1$ and $M_2$ are cographic and not graphic, then Lemma 5.6 and the fact that $M$ is not cographic imply that $M$ has an $R_{15}$-minor, and so since $M$ is minimal, $M \cong R_{15}$. So now assume that $M$ cannot be expressed as a 3-sum of two cographic matroids.

Now if we choose $M_1$ so that $|M_1|$ is minimal, then it must be that $M_1$ is cographic. Now among all possible choices for $M_1$ where $M_1$ is cographic, choose so that $|M_1|$ is maximal. Let $T_2$ be the triangle along which the sum $M_1 \oplus M_2$ is taken. Now let $k \geq 2$ be the maximum-possible integer such that $M = N_1 \oplus N_2 \oplus \ldots \oplus N_k$, where $N_1 = M_1 \cup T_2 \cup \ldots \cup T_k$, the sum of $N_i$ with $N_1$ is along triangle $T_i$, each $r(N_i) > 2$, and $r_{N_1}(T_2 \cup \ldots \cup T_k) = 2$. Note that $N_1$ is still cographic and that by Proposition 2.11, none of $N_1, N_2, \ldots, N_k$ are graphic. In Case 1 say that some $N_i$ for $i \geq 2$ is cographic, and in Case 2 say none of $N_2, \ldots, N_k$ are cographic.

Case 1. Without loss of generality say that $N_2$ is cographic. By assumption $M$ is not a 3-sum of two cographic matroids, and so $k \geq 3$. So we have at least three terms summed along a common line with two being cographic and not graphic. So by Lemma 5.10, we have an $R_{15}$-minor, and by the minimality of $M$ we get $M \cong R_{15}$.

Case 2. Since $N_2$ is neither graphic nor cographic we can write $N_2 = P_1 \oplus P_2$, where $r(P_1) > 2$, $P_1$ sums into $N_1$, and $P_2$ sums into $P_1$. However, by the maximality of $k$, the triangle along which the 3-sum $P_1 \oplus P_2$ is taken, say $T_2'$, must satisfy $r_{P_1}(T_2' \cup T_k) > 2$. Therefore, we can now choose $P_1$ and $P_2$ so that $|P_1|$ is minimal, and we must get that $P_1$ is either graphic or cographic. In Case 2.1 say that $P_1$ is cographic and not graphic, and in Case 2.2 say that $P_1$ is graphic.

Case 2.1. In this case we cannot have that $k = 2$, because then $M = (M_1 \oplus P_1) \oplus P_2$ where, by the minimality of $M$, $M_1 \oplus P_1$ is signed-graphic. So since both $M_1$ and $P_1$ are cographic and not graphic, Lemma 5.7 implies that $M_1 \oplus P_1$ is cographic, which contradicts the maximality of $M_1$. So now that $k \geq 3$ we have that $M_1 \oplus P_1 \oplus N_3$ is a minor of $M$, and these three terms are summed along a common line, and so Lemma 5.10 implies that $M$ has an $R_{15}$-minor and so $M \cong R_{15}$.

Case 2.2. Among all possible choices for $P_1$ and $P_2$ where $P_1$ is graphic, choose so that $|P_1|$ is maximal. Now let $m \geq 2$ be the maximum integer such that $N_2 = Q_1 \oplus Q_2 \oplus \ldots \oplus Q_m$, where $Q_1 = P_1 \cup T_2' \cup \ldots \cup T_m'$, the sum of $Q_1$ with $Q_1$ is along triangle $T'_i$, each $r(Q_i) > 2$, and $r_{Q_1}(T_2' \cup \ldots \cup T_m') = 2$. Note that $Q_1$ is still graphic and, by the maximality of $|P_1|$, none of $Q_2, \ldots, Q_m$ are graphic. In Case 2.2.1 say that at least one of $Q_2, \ldots, Q_m$ is cographic and $k = 2$, in Case 2.2.2 say that at least one of $Q_2, \ldots, Q_m$ is cographic and $k > 2$, and in Case 2.2.3 say that none of $Q_2, \ldots, Q_m$ are cographic.

Case 2.2.1. Without loss of generality say that $Q_2$ is cographic. Now consider $M_1 \oplus P_1 \oplus Q_2$. Since $M_1$ and $Q_2$ are cographic and not graphic we can apply Lemma 5.11.
It cannot be that part (1) holds, because then \( M = M_1 \oplus_3 P_1 \oplus_3 M' \), where \( M_1 \oplus_3 P_1 \) is cographic, contradicting the maximality of \( M_1 \). If part (2) of Lemma 5.11 holds, then \( M = M'' \oplus_3 M(G) \), where \( M'' \) is signed-graphic by the minimality of \( M \), and \( M(G) \) is graphic. But now Lemma 2.11 implies that \( M \) is signed-graphic, a contradiction. Thus \( M \) has an \( R_{15} \) or \( R_{16} \)-minor, and so \( M \cong R_{15} \) or \( R_{16} \).

**Case 2.2.2.** Without loss of generality say that \( Q_2 \) is cographic. Now consider \( (N_1 \oplus_3 N_3 \oplus_3 P_1) \oplus_3 Q_2 \) minus any parallel edges along triangles. Since the terms in \( N_1 \oplus_3 N_3 \oplus_3 P_1 \) are summed along a common line, and since \( Q_2 \) sums into \( P_1 \), which is graphic, we can use Lemma 5.9 to get that \( N_1 \oplus_3 N_3 \oplus_3 Q_2 \) is a 3-connected minor of \( M \), where the three terms are summed along a common line. So now, by Lemma 5.10, we get an \( R_{15} \)-minor of \( N_1 \oplus_3 N_3 \oplus_3 Q_2 \), and so \( M \cong R_{15} \).

**Case 2.2.3.** Since \( Q_2 \) is neither graphic nor cographic, \( Q_2 = R_1 \oplus_3 R_2 \), where \( r(R_1) > 2 \), \( T_1'' \subseteq R_1 \), and by the maximality of \( M \), the triangle along which the 3-sum is taken, call it \( T_2'' \), satisfies \( r_{Q_2}(T_2'' \cup T_1'') > 2 \). As before, we can choose \( R_1 \) so as to minimize \( |R_1| \), which will then make \( R_1 \) either graphic or cographic. In Case 2.2.3.1 say that \( R_1 \) is cographic and not graphic and \( k = 2 \), in Case 2.2.3.2 say that \( R_1 \) is cographic and not graphic and \( k > 2 \), and Case 2.2.3.3 say that \( R_1 \) is graphic.

**Case 2.2.3.1.** Consider the minor \( M_1 \oplus_3 P_1 \oplus_3 R_1 \) of \( M \) and we are done, as in Case 2.2.1.

**Case 2.2.3.2.** Consider the minor \( (N_1 \oplus_3 N_3 \oplus_3 P_1) \oplus_3 R_1 \) of \( M \) and we are done, as in Case 2.2.2.

**Case 2.2.3.3.** Since \( R_1 \) is graphic, we can use Lemma 5.9 on \( R_1 \) to obtain \( N_1'' = Q_1 \oplus_3 Q_3 \oplus_3 \ldots \oplus_3 Q_m \oplus_3 R_2 \) as a minor of \( N_1 = (Q_1 \oplus_3 Q_3 \oplus_3 \ldots \oplus_3 Q_m) \oplus_3 (R_1 \oplus_3 R_2) \), where all terms in the sum for \( N_1'' \) are along a common line. So now \( M'' = N_1 \oplus_3 N_1'' \oplus_3 N_3 \oplus_3 \ldots \oplus_3 N_k \) is a minor of \( M \) and is 3-connected.

Let \( n \geq 1 \) be the largest integer such that \( R_2 = S_1 \oplus_3 \ldots \oplus_3 S_n \), where all terms are summed along a common line parallel to \( T_2'' \) in \( R_2 \) and each \( r(S_i) > 2 \). It cannot be that any \( S_i \) from \( S_1, \ldots, S_n \) is graphic, because then \( M = M' \oplus_3 S_i \) would be signed-graphic by Proposition 2.11 and the minimality of \( M \). So then either there is some \( S_i \) that is cographic or not. In the case that there is some \( S_i \) that is cographic and \( k = 2 \), we get that \( M'' \) has an \( R_{15} \) or \( R_{16} \)-minor in a similar way to Case 2.2.1, and hence \( M \cong R_{15} \) or \( R_{16} \). In the case that some \( S_i \) is cographic and \( k > 2 \), then we get that \( M'' \) has an \( R_{15} \)-minor in a similar way to Case 2.2.2, and hence \( M \cong R_{15} \). In the case that none of \( S_1, \ldots, S_n \) are cographic, we can repeat this process in Case 2.2.3 on \( S_1 \) in \( M'' \). Eventually this process must halt with the conclusion that \( M \cong R_{15} \) or \( R_{16} \).

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References


