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VARIATIONAL PRINCIPLES FOR AVERAGE EXIT TIME MOMENTS FOR DIFFUSIONS IN EUCLIDEAN SPACE

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ABSTRACT. Let $D$ be a smoothly bounded domain in Euclidean space and let $X_t$ be a diffusion in Euclidean space. For a class of diffusions, we develop variational principles which characterize the average of the moments of the exit time from $D$ of a particle driven by $X_t$, where the average is taken over all starting points in $D$.

1. INTRODUCTION

In this note we study diffusions on $\mathbb{R}^d$ and properties of their corresponding exit times from smoothly bounded, connected, open domains in $\mathbb{R}^d$ with compact closure. We will denote by $X_t$ a diffusion in $\mathbb{R}^d$ with corresponding generator $L$ a uniformly elliptic operator of divergence form. We will write $Lf = \text{div}(a\nabla f)$ where the coefficient matrix $a = a_{ij}(x)$ is smooth and symmetric.

Let $\tau = \tau(\omega) = \inf\{t \geq 0 : X_t(\omega) \notin D\}$ be the first exit time of $X_t$ from $D(S)$. We study the average $k$th moment of the exit time for a particle driven by $X_t$, starting in $D$:

$$E_k = E_k(D) = \int_D E_x(\tau^k)dx$$

where $E_x$ denotes expectation under the measure $P_x$ satisfying $P_x\{X_0 = x\} = 1$, for all $x \in \mathbb{R}^d$. Note that $E_k$ is invariant under Euclidean motions.

We give a variational characterization of $E_k$ for each positive integer value of $k$ in the following theorem:

Theorem 1.1. Let $X_t$ be a diffusion on $\mathbb{R}^d$ with generator $L$ a uniformly elliptic operator of divergence form, $Lf = \text{div}(a\nabla f)$, where the coefficient matrix $a$ is smooth and symmetric. Let $D$ be a smoothly bounded open domain in $\mathbb{R}^d$ with compact closure, $\bar{D}$. Define $E_k$ as above and let $F_k$ be defined by

$$F_k = \left\{ f \in C^\infty(\bar{D}); \int_D f(x)dx \neq 0, \ f = Lf = \cdots = L^{k-1}f = 0 \text{ on } \partial D \right\}.$$
Let $\left\lceil \frac{k}{2} \right\rceil$ be the greatest integer of $\frac{k}{2}$. Then, for $k$ even,

$$
\mathcal{E}_k = k! \sup_{f \in \mathcal{F}_k} \frac{(\int_D f)^2}{\int_D |\mathcal{L}^{\frac{1}{2}} f|^2}
$$

and for $k$ odd,

$$
\mathcal{E}_k = k! \sup_{f \in \mathcal{F}_k} \frac{(\int_D f)^2}{\int_D \|
abla L^{\frac{1}{2}} f\|^2_L}
$$

where $\langle \nabla f, \nabla g \rangle_L = \sum_{i,j} a_{ij} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i}$ is the inner product associated with $L$.

The proof of Theorem 1.1 is an application of the generalized Dynkin formula [AK] (cf. also [P1]), followed by an explicit computation. That smooth minimizers for the variational principles cited in Theorem 1.1 exist is explicit in our computations.

Our study of the sequence $\{\mathcal{E}_k\}$ is largely motivated by the now classic work in spectral analysis concerning to what extent a smoothly bounded domain in Euclidean space is determined by its Dirichlet spectrum. More precisely, when the diffusion is standard Brownian motion with generator $L = \frac{1}{2} \Delta$, we are interested in studying to what extent the sequence $\{\mathcal{E}_k\}$ determines the geometry of the underlying domain. There are a number of preliminary results in this direction. For example, in [KMM] the authors prove that among domains of a fixed volume, each element of the sequence is maximized if and only if the underlying domain is a ball of the appropriate volume.

For the case $k = 1$, it is known that the functional $\mathcal{E}_1$ computes the torsional rigidity of a domain. The St. Venant torsion problem, a problem with a long and distinguished history, is to determine those domains of a given volume which maximize torsional rigidity. The problem was settled by Polya (cf. [P2]) who proved that among domains of a fixed volume, the torsional rigidity is maximized by a ball. This result can be recovered using (1.2) and properties of the quotient given in (1.2) under symmetric rearrangement (cf. also [KM1]).

2. Basic results and definitions

Let $(\Omega, \mathcal{B})$ be a measurable space and $\{P_x\}_{x \in \mathbb{R}^d}$ a family of probability measures on $(\Omega, \mathcal{B})$. Let $\{X_t\}_{t \geq 0}$ denote a $d$-dimensional diffusion with generator $L$, a uniformly elliptic operator in divergence form and for which $P_x\{X_0 = x\} = 1$, for $x \in \mathbb{R}^d$.

Let $D$ be a smoothly bounded, connected, open domain with compact closure. As in the introduction, we define the first exit time for a particle driven by $X_t$ from $D$ by $\tau = \tau(\omega) = \inf\{t : X_t(\omega) \notin D\}$. For each $x \in \mathbb{R}^d$, we will denote the expected value of a random variable $Y$ under the probability measure $P_x$ by $E_x(Y)$.

There is a useful relationship between the solution of a certain Poisson problem on the domain $D$ and the expected value of the $k$th power of the first exit time of a particle driven by $X_t$ from $D$ starting at $x \in D$. Suppose $u_k$ solves the problem

$$
L^k u_k + (-1)^{k-1}k! = 0 \text{ on } D,
$$

$$
u_k = Lu_k = \cdots L^{k-1} u_k = 0 \text{ on } \partial D.
$$
Note that $u_k$ can be defined inductively by

$$\begin{align*}
Lu_1 + 1 &= 0 \text{ on } D, \\
u_1 &= 0 \text{ on } \partial D
\end{align*}$$

and

$$\begin{align*}
Lu_k + ku_{k-1} &= 0 \text{ on } D, \\
u_k &= 0 \text{ on } \partial D.
\end{align*}$$

Using the generalized Dynkin formula [H] (cf. also [AK] and [P1]) we have

$\begin{align*}
E_x[u_k(X_0)] - E_x[u_k(X_\tau)] &= \sum_{j=1}^{k-1} \frac{(-1)^j}{j!} E_x[\tau^j L^j u_k(X_\tau)] \\
&\quad + \frac{(-1)^k}{(k-1)!} E_x \left[ \int_0^\tau s^{k-1} L^k u_k(X_s) ds \right].
\end{align*}$

Using the definition of $u_k$ and $\tau$, this gives $u_k(x) = E_x[\tau^k]$ and $E_k$ can be expressed in terms of $u_k$ by $E_k(D) = \int_D u_k(x) dx$.

We will need a number of integral formulae involving the function $u_1$ and the geometry of the diffusion $L$. To ease notation in the sequel we define, for $\alpha$ and $\beta$ tangent vectors at $x \in D$, a scalar product, $\langle \alpha, \beta \rangle_L$, by

$$\langle \alpha, \beta \rangle_L = \alpha^T a(x) \beta$$

where $\alpha^T$ denotes the transpose of $\alpha$.

Let $u_1$ be as defined in (2.1) and let $f \in \mathcal{F}_k$. Let $\nu$ be the outward pointing unit normal vector to $\partial D$. By the Divergence Theorem,

$$\int_D f Lu_1 - u_1 Lf = \int_{\partial D} f (\nabla u_1, \nu)_L - u_1 (\nabla f, \nu)_L = 0.$$ 

We conclude

$$\int_D f = - \int_D u_1 Lf.$$ 

If $X$ is a vectorfield on $D$ and $f \in \mathcal{F}_k$, then $\text{div}(fX) = f \text{div}(X) + \langle \nabla f, X \rangle$ where $\langle \alpha, \beta \rangle$ is the standard scalar product. By the Divergence Theorem,

$$\int_D \text{div}(fX) = \int_{\partial D} f \langle X, \nu \rangle = 0$$

and we conclude that

$$\int_D f \text{div}(X) = - \int_D \langle \nabla f, X \rangle.$$ 

In particular, if $u_1$ is as defined in (2.1) and $X = a_{ij}(x) \nabla u_1$, then

$$\int_D f = \int_D \langle \nabla u_1, \nabla f \rangle_L.$$
3. Variational characterizations

Throughout this section let \( D \) be as above and let \( \mathcal{F}_k \) be given as in Theorem 1.1. We begin with a lemma which generalizes (2.4) and (2.5).

**Lemma 3.1.** Let \( u_n \) be as defined by (2.2), let \( k \) be a positive integer, and let \( f \in \mathcal{F}_k \). If \( k = 2n \), then

\[
\int_D f = \frac{(-1)^n}{n!} \int_D u_n L^nf.
\]

If \( k = 2n + 1 \), then

\[
\int_D f = \frac{(-1)^n}{(n + 1)!} \int_D (\nabla u_{n+1}, \nabla L^nf)_L
\]

where the scalar product is as given in (2.3).

**Proof.** Suppose that \( k = 2n \) and for \( 0 < l < n - 1 \), define

\[
P_l = (L^lu_n)(L^{n-l}f) - (L^{l+1}u_n)(L^{n-(l+1)}f).
\]

Then

\[
\sum_{l=0}^{n-1} P_l = u_n L^nf - f L^nu_n.
\]

Let \( \nu \) be the outward pointing unit normal vector along \( \partial D \). By the Divergence Theorem and the fact that \( L^lu_n = 0 \) on \( \partial D \), for \( l = 0, \ldots, n - 1 \),

\[
\int_D P_l = \int_{\partial D} (L^lu_n)(\nabla L^{n-l-1}f, \nu)_L - (L^{n-(l+1)}f)(\nabla L^lu_n, \nu)_L = 0.
\]

Combining (3.3) and (3.4) and using that \( L^n u_n = (-1)^n n! \), we have established (3.1).

Suppose \( k = 2n + 1 \) and for \( 0 \leq l \leq n \), define

\[
R_l = (L^lu_{n+1})(L^{n+1-l}f) - (L^{l+1}u_{n+1})(L^{n+1-(l+1)}f).
\]

Then

\[
\sum_{l=0}^{n} R_l = u_{n+1} L^{n+1}f - f L^{n+1}u_{n+1}.
\]

As above, we use the Divergence Theorem to see that

\[
\int_D R_l = \int_{\partial D} (L^lu_{n+1})(\nabla L^{n-l}f, \nu)_L - (L^{n-l}f)(\nabla L^lu_{n+1}, \nu)_L = 0.
\]

Since \( L^{n+1}u_{n+1} = (-1)^{n+1}(n + 1)! \), we conclude

\[
\int_D f = \frac{(-1)^{n+1}}{(n + 1)!} \int_D u_{n+1} L(L^n f).
\]

If \( X \) is the vectorfield given by \( X = a \nabla (L^n f) \), then following the argument used to establish (2.5),

\[
\int_D u_{n+1} L(L^n f) = \int_D u_{n+1} \text{div}(X)
\]

\[
= -\int_D (\nabla u_{n+1}, \nabla L^n f)_L
\]

and we have established (3.2). \( \square \)
We now prove Theorem 1.1. Suppose $k = 2n$ and, for $f \in \mathcal{F}_k$, consider the quotient

$$Q_k(f) = \frac{(\int_D f)^2}{\int_D |L^n f|^2}.$$  

From (3.1)

$$Q_k(f) = \left( \frac{1}{n!} \right)^2 \frac{(\int_D u_n L^n f)^2}{\int_D |L^n f|^2}.$$  

Let $G_k = \{ g \in \mathcal{F}_n : g = L^n f \text{ for some } f \in \mathcal{F}_k \}$. Let $\mathcal{H}_k$ be the completion of $G_k$ in the Hilbert space, $\mathcal{L}^2$, of square integrable functions on $D$. If we denote the inner product of $g$ and $h \in \mathcal{L}^2$ by $\langle g, h \rangle$ and by $\|g\|$ the $L^2$ norm of $g$, then we can view $Q_k$ as a map $Q_k : G_k \subset \mathcal{H}_k \rightarrow \mathbb{R}$,

$$Q_k(g) = \left( \frac{1}{n!} \right)^2 \left( \frac{\langle u_n, g \rangle}{\|g\|} \right)^2.$$  

Clearly, the domain of $Q_k$ can be extended to nonzero elements of $\mathcal{H}_k$ and $Q_k(cg) = Q_k(g)$ for every nonzero scalar $c$. It follows that $Q_k$ is maximized when $g \in \mathcal{H}_k$ is in the direction of $u_n \in \mathcal{H}_k$. If $g = cu_n$ we have $L^n(c' u_{2n}) = g$, and computing $Q_k(cu_n)$ we see that

$$\sup_{g \in \mathcal{H}_k} Q_k(g) = Q_k(cu_n) = \frac{(\int_D u_{2n})^2}{\int_D (L^n u_{2n})^2},$$  

where we have applied (3.1) of Lemma 3.1 to the numerator. Note that $(L^n u_{2n})^2 = (-1)^n (2n)! u_n L u_{2n}$. Applying Lemma 3.1 to the denominator we obtain

$$\sup_{g \in \mathcal{H}_k} Q_k(g) = \frac{(\int_D u_{2n})^2}{(2n)! \int_D u_{2n}} = \frac{1}{k!} \mathcal{E}_k(D),$$  

which establishes (1.1) of Theorem 1.1.

The proof of (1.2) of Theorem 1.1 is similar. Suppose $k = 2n+1$ and, for $f \in \mathcal{F}_k$, consider the quotient

$$\tilde{Q}_k(f) = \frac{(\int_D f)^2}{\int_D |\nabla L^n f|^2}.$$  

From (3.2) of Lemma 3.1,

$$\tilde{Q}_k(f) = \left( \frac{1}{(n+1)!} \right)^2 \frac{(\int_D \langle \nabla u_{n+1}, \nabla L^n f \rangle_L)^2}{\int_D |\nabla L^n f|^2_L}.$$  

Let $C^\infty(\tilde{D}, \mathbb{R}^d)$ be the space of smooth vectorfields on $\tilde{D}$. Let

$$\tilde{G}_k = \{ X \in C^\infty(\tilde{D}, \mathbb{R}^d) : X = \nabla g \text{ for some } g \in \mathcal{F}_{n+1} \} \quad \text{with } g = L^n f \text{ for some } f \in \mathcal{F}_k \}.$$
Let $\tilde{\mathcal{H}}_k$ be the completion of $\tilde{\mathcal{G}}_k$ in the space of vector fields square integrable with respect to the inner product $\langle \alpha, \beta \rangle_L$. We can view $\tilde{Q}_k$ as a map $\tilde{Q}_k : \tilde{\mathcal{H}}_k \to \mathbb{R}$,

$$
\tilde{Q}_k(g) = \left( \frac{1}{(n+1)!} \right)^2 \frac{\left( \langle \nabla u_{n+1}, g \rangle_L \right)^2}{\|g\|^2_L}.
$$

It is clear that the domain of $\tilde{Q}_k$ extends to nonzero vectors in the space $\tilde{\mathcal{H}}_k$ and that for all nonzero scalars $c$, $\tilde{Q}_k(cg) = \tilde{Q}_k(g)$. It follows that $\tilde{Q}_k$ is maximized when $g = c\nabla u_{n+1}$ where $c$ is some nonzero constant. Computing $\tilde{Q}_k(\nabla u_{n+1})$ we see that

$$
\sup_{g \in \tilde{\mathcal{H}}_k} \tilde{Q}_k(g) = \tilde{Q}_k(c\nabla u_{n+1})
$$

$$
= \frac{(\int_D u_{2n+1})^2}{\int_D \|\nabla L^n u_{2n+1}\|^2_L},
$$

where we have used (3.2) on the numerator.

Note that $\|\nabla L^n u_{2n+1}\|^2_L = \left( \frac{1}{(n+1)!} \right)^2 (2n+1)! \langle \nabla u_{n+1}, \nabla L^n u_{2n+1} \rangle_L$. Applying (3.2) of Lemma 3.1 to the denominator we obtain

$$
\sup_{g \in \tilde{\mathcal{H}}_k} \tilde{Q}_k(g) = \frac{(\int_D u_{2n+1})^2}{(2n+1)! \int_D u_{2n+1}}
$$

$$
= \frac{1}{k!} \mathcal{E}_k(D)
$$

which establishes (1.2) of Theorem 1.1.

REFERENCES


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