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On the Existence of Multiple Positive Solutions for a Semilinear Problem in Exterior Domains *

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July 12, 1999

Abstract

In this paper, we study the existence and nonexistence of multiple positive solutions for problem

\[
\begin{cases}
\Delta u + K(x)u^p = 0 & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \quad u \in H^1_{\text{loc}}(\Omega) \cap C(\overline{\Omega}), \\
u|_{\partial \Omega} = 0, & \quad u \to \mu > 0 \quad \text{as } |x| \to \infty
\end{cases}
\]

where \( \Omega = \mathbb{R}^N \setminus \omega \) is an exterior domain in \( \mathbb{R}^N \), \( \omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary and \( N > 2, \mu \geq 0, p > 1 \) are some given constants. \( K(x) \) satisfies:

\( K(x) \in C^\alpha_{\text{loc}}(\Omega) \) and \( \exists C, \epsilon, M > 0 \) such that, \( |K(x)| \leq C|x|^{\epsilon} \) for any \( |x| \geq M \), with

*Research supported by the Natural Science Foundation of China
$l \leq -2 - \epsilon$. Some existence and nonexistence of multiple solutions have been discussed under different assumptions on $K$.

Key words and phrases: multiple solutions, critical exponents, elliptic equations.

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AMS Classification: 35J10 35J20 35J60 35J65

1 Introduction

In this paper, we study the existence of multiple solutions for problem

$$
\begin{aligned}
\Delta u + K(x)u^p &= 0 \quad \text{in } \Omega, \\
u &> 0 \quad \text{in } \Omega, \quad u \in H^1_{\text{loc}}(\Omega) \cap C(\overline{\Omega}). \\
u|_{\partial \Omega} &= 0,
\end{aligned}
$$

(1.1)

with the boundary condition $u \to \mu > 0$ as $|x| \to \infty$, where $\Omega = \mathbb{R}^N \setminus \omega$ is an exterior domain in $\mathbb{R}^N$, $\omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N > 2$. $p > 1$ is a given constants. $K(x)$ satisfies:

(H$_1$) $K(x) \in C^0_{\text{loc}}(\Omega)$, $K \not\equiv 0$ and $\exists C, \epsilon, M > 0$ such that, $|K(x)| \leq C|x|^l$ for any $|x| \geq M$,

with $l \leq -2 - \epsilon$.

Such a problem occur in various branches of mathematical physis and Geometry. For $K(x) \equiv |x|^l$, $\Omega = \mathbb{R}^N$ equation (1.1) is known as Lane-Emden equation, sometimes it is also referred to as the Emden-Fowler equation in astrophysics, where $u$ represents the density of a single star. When $p = \frac{N+2}{N-2}$, $\Omega = \mathbb{R}^N$ and $n \geq 3$, equation (1.1) is called the conformal scalar curvature equation in $\mathbb{R}^N$. Let $g$ be the usual metric in $\mathbb{R}^N$, the problem of finding a metric $g_1$ which is conformal to $g$ (i.e. $g_1 = u^{\frac{4}{N-2}}g$, for some positive function $u$ with scalar curvature $\tilde{K}$ is equivalent to that to find a positive solution of (1.1) with $K = \frac{N-2}{4(N-1)}\tilde{K}$. For a detail overview on (1.1), we refer readers to the papers [N2], [LN1], [Z] and the references therein.
1 INTRODUCTION

Equations like (1.1) has been studied by many mathematicians ([B], [CZ], [CL1-2], [DL1-2], [DLZ], [DN1-2], [Es], [G], [GE], [JPY], [KL], [LY1-2], [Lio], [WW], [Y], [YY], [ZC]). Ni ([N1]), Kenig and Ni ([KN]) proved existence theorems for (1.1) under the condition $(H_1)$. It is shown in ([N1]) that if $K$ is nonnegative with $K \geq Cr^l$ for some $l > (N - 2)(p - 1) - 2$ at infinity, or if $K$ is nonpositive with $-K \geq Cr^l$ for some $l > -2$ at infinity then (1.1) possesses no positive solutions, where $C > 0$. Lin ([Lin]) proved the existence for (1.1) under the condition that $|K| \leq \varphi(|x|^2)$ at infinity with $\int_{\infty}^{r} \varphi(r) dr < \infty$. Lin in [Lin] also proved a nonexistence result when $K$ is nonpositive with $-K \geq Cr^{-2}$ at infinity. Other nonexistence results are given in [BLY] and [LN1]. In case of that $|K| \leq C r^{(N-2)(p-1)-2-\varepsilon}$ at infinity for some positive constants $C$ and $\varepsilon$, the existence and asymptotics of positive solutions are studied by many authors, here we only mention the results of, for example, Ni, Yosutani [NY], [LN1], LN2 and Li [L2]. In the fast decay case $|K| \leq Cr^l$, $l < -2$, Ni showed that (1.1) possesses infinitely many positive solutions which are bounded from below by positive constants (see [N1] and [LN1]). Li and Ni ([LN1]) showed that, for positive bounded solution of (1.1), the limit $u_\infty = \lim_{x \to \infty} u(x)$ always exists for any $\varepsilon > 0$, furthermore, if $u_\infty = 0$, then

$$u(x) \leq \begin{cases} C|x|^{2-N} & \text{if } p > \frac{N+l}{N-2}, \\ C\varepsilon |x|^{\frac{(1-\varepsilon)(l+2)}{l-p}} & \text{if } p \leq \frac{N+l}{N-2}, \end{cases}$$

and if $u_\infty > 0$, then

$$|u - u_\infty| \leq \begin{cases} C|x|^{2-N} & \text{if } l < -N, \\ C|x|^{2-N} \log |x| & \text{if } l = -N, \\ C|x|^{2+l} & \text{if } -N < l < -2, \end{cases}$$

at $\infty$. These results are refined in [LN2] and [L2].

Recently, Zhao (see [Z]) studied the following problem:

$$\begin{cases} \Delta u + K(x)f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega, \quad u \in H^1_{\text{loc}}(\Omega) \cap C(\overline{\Omega}) \\ u|_{\partial \Omega} = 0, & u \to \mu > 0 \text{ as } |x| \to \infty. \end{cases} \quad (1.1)_\mu$$
1 INTRODUCTION

The existence of one positive solution of problem (1.1)_{\mu} when \( f \) is superlinear at 0 was obtained with some assumptions (Green-tight function) on \( K(x) \) for small \( \mu > 0 \). A natural and interesting problem is that how many solutions can be obtained for a given \( \mu > 0 \). There seems to have been little progress in this direction. The purpose of this paper is to discuss the existence and nonexistence of multiple solutions for problem (1.1)_{\mu} for a given \( \mu > 0 \).

The main results of this paper can be included in the following theorems:

**Theorem 1.1.** Suppose \((H_1)\). Let \( h(x) \) be a positive harmonic function in \( \Omega \) satisfying

\[
\begin{align*}
  &h(x) \bigg|_{\partial \Omega} = 0, \\
  &\lim_{|x| \to \infty} h(x) = 1.
\end{align*}
\]

Then

(i) If \( K(x) \leq 0 \), then for any \( \mu > 0 \), there exists a unique solution \( u_{\mu} \) of (1.1)_{\mu}. In addition, 
\( u_{\mu} \leq \mu h \) on \( \Omega \) and \( u_{\mu} \) is increasing with respect to \( \mu \).

(ii) If \( K(x) \geq 0 \), \( \exists \mu^* \in (0, -\infty) \) and \( \mu^* < +\infty \), such that for \( \mu > \mu^* \) there does not exist a solution of (1.1)_{\mu}; and for \( \mu \in (0, \mu^*) \), there exists a minimal solution \( u_{\mu} \) of (1.1)_{\mu}. In addition, \( u_{\mu} \) is increasing with respect to \( \mu \), \( u_{\mu} \geq \mu h \) in \( \Omega \) and, as \( \mu \to \mu^* \), \( u_{\mu} \) increases to \( u_{\mu^*} \) the minimal solution of (1.1)_{\mu^*}, and \( u_{\mu^*} \) is unique.

(iii) If \( K(x) \) change sign, we can find a \( \mu^* \in (0, +\infty) \) such that problem (1.1)_{\mu} possesses at least one solution for all \( \mu \in (0, \mu^*) \).

**Theorem 1.2.** Suppose that \( p = \frac{N+2}{N-2} \), \((H_1)\), \( 0 \leq K(x) \in L^1(\Omega) \), and

\((H_2)\) \( K(x) > 0 \) in a neighborhood \( V \) of some point \( x_0 \in \Omega \) such that

\[
K(x_0) = \sup_{x \in \Omega} K(x)
\]

and \( K(x) = K(x_0) + O(|x-x_0|^2) \) near \( x_0 \). Then problem (1.1)_{\mu} possesses at least two solutions \( u_{\mu} \) and \( U_{\mu} \) with \( u_{\mu} < U_{\mu} \) if \( \mu \in (0, \mu^*) \), where \( \mu^* \) is given by Theorem 1.1.

This paper is organized as follows: We first give some Lemmas in Section 2, which will be used in the proof of Theorem 1.1. Then the existence and nonexistence of minimal solution for problem (1.1)_{\mu} is given in Section 3 by the standary barrier method. Finally, the existence of the second solution for (1.1)_{\mu} is given in Section 4 by using the variational method.
2 Preliminaries

In this Section, we will prove some Lemmas which will be used in the proof of Theorem 1.1.

Lemma 2.1. Let $f$ be a locally Hölder continuous function on $\Omega$ with the following decay property

$$|f(x)| \leq C|x|^{\ell} \quad \text{at } \infty$$

with $C > 0$, $\ell < -2$, and $w$ be the Newtonian potential of $f$, i.e.

$$w(x) = \int_{\Omega} G(x, y) f(y) dy .$$

where $G(x, y)$ is the Green function for $\Omega$ corresponding to the Laplacian $-\Delta$. Then $w(x)$ is well-defined and at $\infty$ we have

$$|w(x)| \leq \begin{cases} 
C|x|^{2-N} & \text{if } \ell < -N \\
C|x|^{2-N} \ln |x| & \text{if } \ell = -N \\
C|x|^{2+\ell} & \text{if } -N < \ell < -2
\end{cases}$$

Proof. This Lemma may be proved by standard arguments. We include a proof here for the sake of completeness.

From the definition of Green function, we can easily deduce that

$$G(x, y) \leq \frac{C_N}{|x - y|^{N-2}} .$$

where $C_N = (N(N-2) \omega_N)^{-1}$ and $\omega_N$ is the volume of the unit ball in $\mathbb{R}^N$. Using this fact and (2.1) we can find a constant $C > 0$ such that

$$|w(x)| \leq C \int_{\Omega} \frac{1}{|x - y|^{N-2}(1 + |y|^{-\ell})} dy .$$

Thus $w(x)$ is well-defined. Next we decompose the integral (2.4) as follows.

$$|w(x)| \leq \left( \int_{|y - x| \leq |x|} + \int_{|y - x| \leq 2|x|} + \int_{|y - x| \leq 2|x|} \right) \frac{C}{|y - x|^{N-2}(1 + |y|^{-\ell})} dy$$

$$\equiv I_1 + I_2 + I_3$$
where $I_1, I_2$ and $I_3$ are defined by the last equality. Same as [LN2] we can conclude that

$$I_1 \leq \frac{C}{|x|^{-\ell}} \int_0^{\frac{|x|}{2}} \frac{1}{r^{N-2}} r^N dr = C|x|^{2+\ell},$$

$$I_2 \leq C \int_{|x|/2}^{2|x|} \frac{1}{r^{N-2}} r^N dr = C|x|^{2+\ell},$$

$$I_3 \leq \begin{cases} C|x|^{2-N} & \text{if } N + \ell < 0, \\ C|x|^{2-N}(\ell n|x| + 1) & \text{if } N + \ell = 0, \\ C|x|^{2-N}(1 + |x|^{N+\ell}) & \text{if } N + \ell > 0, \end{cases}$$

Now, it is easy to see that (2.2) holds.

**Lemma 2.2.** Under the assumption of Lemma 2.1, suppose $v$ is a solution of

$$\begin{cases} -\Delta v = f(x) & \text{in } \Omega, \\ v|_{\partial \Omega} = 0 & \lim_{|x| \to \infty} v(x) = \mu. \end{cases} \quad (2.5)$$

Then

$$v = \mu h(x) + \int_{\Omega} G(x,y) f(y) dy \quad (2.6)$$

where $h(x)$ is the positive harmonic function in $\Omega$ satisfying

$$h(x)|_{\partial \Omega} = 0, \quad \lim_{|x| \to \infty} h(x) = 1 \quad (2.7)$$

and $G(x,y)$ is the Green functions for $\Omega$ corresponding to $\Delta$.

**Proof.** From [Z] and Lemma 2.1, we can deduce that $h(x)$ exists with $0 < h < 1$ in $\Omega$ and the integral in (2.6) is well-defined. Set $w(x) = \int_{\Omega} G(x,y) f(y) dy$. For an arbitrary but fixed point $z \in \Omega$, choose $R$ large enough such that $R > |z|$ and $\omega \subset B_R(0)$. Now we define

$$w_1(x) = \int_{\Omega \cap B_R(0)} G(x,y) f(y) dy,$$

$$w_2(x) = \int_{\mathbb{R}^N \setminus B_R(0)} G(x,y) f(y) dy.$$

Then it is standard that

$$\Delta w_1(z) + f(z) = 0 \text{ and } \Delta w_2(z) = 0.$$
2 PRELIMINARIES

Since \( w = w_1 + w_2 \) we have
\[
\Delta w + f = 0 \quad \text{in } \Omega. \tag{2.8}
\]

By Lemma 2.1 and the property of Green functions we have
\[
w \bigg|_{\partial \Omega} = 0, \quad \lim_{|x| \to \infty} w(x) = 0. \tag{2.9}
\]

Therefore
\[
\begin{cases}
\Delta(v - w) = 0, \\
(v - w) \big|_{\partial \Omega} = 0, \quad \lim_{|x| \to \infty}(v - w) = \mu.
\end{cases}
\]

By the uniqueness of the above problem, we have
\[
v - w = \mu h(x). \]

This gives (2.6).

\[\square\]

**Theorem 2.3.** Suppose \((H_1)\) and let \( u \) be a bounded solution of \((1.1)_\mu\), then
\[
|u(x) - \mu h(x)| \leq \begin{cases}
C|x|^{2-N} \text{ at } \infty & \text{if } \ell < -N \\
C|x|^{2-N} \ell n|x| \text{ at } \infty & \text{if } \ell = -N \\
C|x|^{2+\ell} \text{ at } \infty & \text{if } -N < \ell < -2
\end{cases}
\]

where \( h(x) \) is the unique solution of (2.7).

The proof of the above theorem can come directly from Lemma 2.1 and Lemma 2.2.

**Lemma 2.4.** Suppose \((H_1)\) with \( l = -\frac{N+2}{2} - \epsilon \) for some \( \epsilon > 0 \), \( K(x) \geq 0 \), \( K(x) \neq 0 \) and \( u_\mu \) be the solution of \((1.1)_\mu\). Then
\[
u_\mu(x) - \mu h(x) \in \mathcal{D}^{1,2}_0(\Omega)
\]

where \( h(x) \) is the unique solution of (2.7) and \( \mathcal{D}^{1,2}_0(\Omega) \) is a Sobolev’s space defining as the completion of \( C_0^\infty(\Omega) \) in the norm \( \int_\Omega |\nabla u|^2 \, dx = \|u\|^2 \).

**Proof.** From Theorem 2.3, (2.7), and (1.1)_\mu we can easily conclude that
\[
\begin{cases}
\Delta(u_\mu - \mu h(x)) + K(x)u_\mu^p = 0 \\
(u_\mu - \mu h(x))|_{\partial \Omega} = 0, \quad \lim_{|x| \to \infty}(u_\mu - \mu h(x)) = 0
\end{cases}
\]
and
\[- \int_{\Omega} \triangle (u_\mu - \mu h(x))(u_\mu - \mu h(x)) \, dx = \int_{\Omega} |\nabla (u_\mu - \mu h(x))|^2 \, dx.\]

Thus
\[
\int_{\Omega} |\nabla (u_\mu - \mu h(x))|^2 \, dx = \int_{\Omega} K(x)u_\mu^p(u_\mu - \mu h(x)) \, dx \\
= \int_{\Omega \cap BR} K(x)u_\mu^p(u_\mu - \mu h(x)) \, dx + \int_{R^N \setminus BR} K(x)u_\mu^p(u_\mu - \mu h(x)) \, dx \\
\leq C + C_1 \int_{R^N} r^l s(r) r^{N-1} \, dr \\
\leq +\infty
\]
if \( l < -\frac{N+2}{2} \). Here
\[
s(r) = \begin{cases} 
|r|^{2-N} & \text{if } l < -N, \\
|r|^{2-N} \ln |r| & \text{if } l = -N, \\
|r|^{2+l} & \text{if } -N < l < -2.
\end{cases}
\]

Remark 2.1. The conclusion of Lemma 2.4 still remain true if we replace the assumption \( l = -\frac{N+2}{2} - \epsilon \) by \( S(|x|)K(x) \in L^1 \) near \( \infty \).

\[ \square \]

3 Existence of minimal solution

In this Section, we will give a complete proof of Theorem 1.1 by the standary barrier method.

Lemma 3.1. Suppose \((H_1)\) and \(K(x) \geq 0, K(x) \not\equiv 0\), then there exists a constant \(0 < \mu^* < \infty\) such that problem \((1.1)_\mu\) possesses a minimal solution for all \(\mu \in (0, \mu^*)\) and no solution for problem \((1.1)_\mu\) for \(\mu > \mu^*\).

Proof. First of all, we prove that problem \((1.1)_\mu\) has a minimal solution if \(\mu\) is small enough.

In fact, let \(\varphi(x) = h(x) + \int_{\Omega} G(x,y)K(y) \, dy\). From Lemma 2.2, \(\varphi(x)\) is a solution of

\[
\begin{cases}
-\Delta \varphi = K(x) & \text{in } \Omega \\
\varphi|_{\partial \Omega} = 0, \lim_{|x| \to \infty} \varphi(x) = 1
\end{cases}
\]
Denoting $\varphi_\mu(x) = \mu \varphi(x)$, we have $\varphi_\mu(x) \geq \mu h(x)$ because $K(x) \geq 0$ in $\Omega$. Then
\[
\begin{cases}
-\Delta \varphi_\mu - K(x)\varphi_\mu^p = K(x)(\mu - (\mu \varphi)^p) \geq 0 \\
\varphi_\mu|_{\partial \Omega} = 0, \quad \lim_{|x| \to \infty} \varphi_\mu = \mu
\end{cases}
\]
if $\mu$ is small enough. So $\bar{u} = \mu \varphi$ is a supersolution of (1.1)$_\mu$ if $\mu$ is small enough. It is easy to check that $\underline{u} = \mu h(x)$ is a subsolution of (1.1)$_\mu$ for all $\mu > 0$ and all positive supersolution of (1.1)$_\mu$ must be larger than or equal to $\mu h$. The method of sub and supersolution yields our first claim.

Next, we set

$$
\mu^* = \sup\{\mu > 0, \text{ | problem (1.1)$_\mu$ possesses at least one solution}\}
$$

(3.2)

so that $\mu^* > 0$. For any $\mu \in (0, \mu^*)$, from the definition of $\mu^*$, we can find an $\bar{\mu} > \mu$ such that problem (1.1)$_{\bar{\mu}}$ possesses a solution $u_{\bar{\mu}}$ and hence $u_{\bar{\mu}}$ is a supersolution of (1.1)$_\mu$. It is easy to verify that $\underline{u}_{\mu} = \mu h(x)$ is a subsolution of (1.1)$_\mu$ for all $\mu > 0$ and all positive supersolution of (1.1)$_\mu$ must be larger than or equal to $\mu h$. Using monotone iteration we can get the minimal solution $u_{\mu}$ for all $\mu \in (0, \mu^*)$.

Now, we are going to prove that $\mu^* < +\infty$. In fact, if $u_{\mu}$ solves (1.1)$_\mu$, since $u_{\mu} \geq \mu h$ we have
\[
\begin{cases}
-\Delta(u_{\mu} - \mu h(x)) = -\Delta u_{\mu} = K(x)(u_{\mu})^{p-1} \geq K(x)(\mu h)^{p-1}(u_{\mu} - \mu h(x)) \text{ in } \Omega \\
(u_{\mu} - \mu h(x)) > 0 \text{ in } \Omega, \\
(u_{\mu} - \mu h(x)) \in \mathcal{D}^{1,2}_0(\Omega)
\end{cases}
\]
Thus the first eigenvalue of $-\Delta - K(x)(\mu h)^{p-1}$ on $\mathcal{D}^{1,2}(\Omega)$ is positive and this is impossible for $\mu$ large.

From the definition of $\mu^*$ we know that there is no solution for problem (1.1)$_\mu$ if $\mu > \mu^*$.

Lemma 3.2. Suppose $H_1$) with $l = -\frac{N+2}{2} - \epsilon$ and $K(x) \geq 0$, $K(x) \neq 0$. $u_{\mu}$ be the minimal solution of (1.1)$_\mu$ for $\mu \in (0, \mu^*)$. Then the minimizing problem
\[
\sigma_\mu = \inf \left\{ \int_\Omega |\nabla w|^2 dx \mid w \in \mathcal{D}^{1,2}_0(\Omega), \quad \int_\Omega pK(x)w^{p-1}w^2 dx = 1 \right\}
\]

(3.3)
can be attained by a function $\psi_\mu > 0$ which satisfies the equation

\[
\begin{cases}
-\Delta w = \sigma pK(x)w^{p-1}_\mu w & \text{in } \Omega \\
w \in D^{1,2}_0(\Omega)
\end{cases}
\] (3.4)$_\mu$

with $\sigma = \sigma_\mu$. Furthermore, $\sigma_\mu > 1$ for all $\mu \in (0, \mu^*)$.

Proof. We first prove that the functional $\int_\Omega pKu^{p-1}_\mu w^2dx$ is weakly sequentially compact. In fact, let $\{w_n\}$ is a bounded sequence in $D^{1,2}_0(\Omega)$ with weak limit $w \in D^{1,2}_0(\Omega)$, the boundedness of $K$ and $u_\mu$ in $\Omega$ and the use of Hölder inequality in a ball $B_R$ for a large $R$, and $B'_R = \mathbb{R}^N \setminus B_R$ give

\[
\int_\Omega Kw^{p-1}_\mu |w_n - w|^2dx \leq C_1 \int_{B'_R \cap \Omega} |w_n - w|^2dx + C \left( \int_{B'_R} |w_n - w|^\frac{2N}{N-2}dx \right)^\frac{N-2}{N} \left( \int_{B'_R} K(x)\frac{N}{2}dx \right)^\frac{2}{N}
\]

where $C, C_1$ are positive constants, independent of $w_n, w$. It follows from the compactness of the embedding $D^{1,2}_0(\Omega \cap B_R) \hookrightarrow L^2(\Omega \cap B_R)$ and assumption (H$_1$) we have

\[
\int_\Omega Kw^{p-1}_\mu (w_n - w)^2dx \leq C_1 \int_{B'_R \cap \Omega} |w_n - w|^2dx + C \int_\Omega r^{-(2+\epsilon)\frac{N}{2}}dx = \epsilon_1 + \epsilon_1 = \epsilon_1
\]

for any $\epsilon_1 > 0$ if $R$ and $n$ are large enough. This gives us that the functional $\int_\Omega pKu^{p-1}_\mu w^2dx$ is weakly sequentially compact. Consequently standard minimization procedure shows that $\sigma_\mu$ is attained by a function $\psi_\mu \geq 0$, $\psi_\mu \in D^{1,2}_0(\Omega)$, satisfying (3.4)$_\mu$ with $\sigma = \sigma_\mu$. By assumption (H$_1$) we deduce $\sigma_\mu pK(x)w^{p-1}_\mu(x) |x|^\delta \in L^q(\Omega)$ for some $\delta > 0$ and $q > \frac{N}{2}$. Therefore a result of Egnel [E] implies that $\psi_\mu$ is bounded in $\Omega$ and $\psi_\mu = 0(|x|^{2-N})$ as $|x| \rightarrow \infty$ and standard Hölder estimates then imply that $\psi_\mu \in C^{3,\alpha}_{\text{loc}}(\Omega)$ for all $0 < \alpha < 1$.

Next, we prove $\sigma_\mu > 1$. In fact, for $\mu < \bar{\mu}$, $\mu, \bar{\mu} \in (0, \mu^*)$ problem (1.1)$_\mu$ and (1.1)$_{\bar{\mu}}$ have a minimal solution $u_\mu$ and $u_{\bar{\mu}}$ respectively. Because $u_{\bar{\mu}}$ is a supersolution of (1.1)$_{\mu}$, we have
$u_{\mu} \leq u_{\bar{\mu}}$. Set $v_{\bar{\mu}} = u_{\bar{\mu}} - \bar{\mu}h$, $v_{\mu} = u_{\mu} - \mu h$. From Lemma 2.5 we have

$$\begin{cases}
-\Delta v_{\bar{\mu}} = K(x)(v_{\bar{\mu}} + \bar{\mu}h)^p & v_{\bar{\mu}} > 0 \text{ in } \Omega \\
v_{\bar{\mu}} \big|_{\partial \Omega} = 0, \lim_{|x| \to \infty} v_{\bar{\mu}}(x) = 0 \text{ and } v_{\bar{\mu}} \in \mathcal{D}^{1,2}_0(\Omega)
\end{cases}$$

and

$$\begin{cases}
-\Delta v_{\mu} = K(x)(v_{\mu} + \mu h)^p & v_{\mu} > 0 \text{ in } \Omega \\
v_{\mu} \big|_{\partial \Omega} = 0, \lim_{|x| \to \infty} v_{\mu}(x) = 0 \text{ and } v_{\mu} \in \mathcal{D}^{1,2}_0(\Omega)
\end{cases}$$

and

$$-\Delta(v_{\bar{\mu}} - v_{\mu}) = K(x)[(v_{\bar{\mu}} + \bar{\mu}h)^p - (v_{\mu} + \mu h)^p] = K(x)(u_{\bar{\mu}}^p - u_{\mu}^p) \geq 0.$$ 

Maximum principle gives us that

$$v_{\bar{\mu}} - v_{\mu} > 0 \text{ in } \Omega. \quad (3.5)$$

Furthermore,

$$\begin{cases}
-\Delta(v_{\bar{\mu}} - v_{\mu}) = K(x)(u_{\bar{\mu}}^p - u_{\mu}^p) \geq K(x)pu_{\mu}^{p-1}(v_{\bar{\mu}} - v_{\mu} + (\bar{\mu} - \mu)h) \\
(v_{\bar{\mu}} - v_{\mu}) \in \mathcal{D}^{1,2}_0(\Omega)
\end{cases} \quad (3.6)$$

On the other hand,

$$\begin{cases}
-\Delta \psi_{\mu} = \sigma_{\mu}K(x)pu_{\mu}^{p-1}\psi_{\mu} & \psi_{\mu} \geq 0 \text{ in } \Omega \\
\psi_{\mu} \in \mathcal{D}^{1,2}_0(\Omega)
\end{cases} \quad (3.7)$$

Multiplying (3.6) by $\psi_{\mu}$ and (3.7) by $w \equiv u_{\bar{\mu}} - u_{\mu}$ we deduce

$$\int_{\Omega} \nabla w \nabla \psi_{\mu} dx \geq \int_{\Omega} pK(x)w_{\mu}^{p-1}(w + (\bar{\mu} - \mu)h)\psi_{\mu} dx$$

and

$$\int_{\Omega} \nabla \psi_{\mu} \nabla w dx = \sigma_{\mu}p \int_{\Omega} K(x)w_{\mu}^{p-1}\psi_{\mu} w dx.$$ 

Thus

$$\sigma_{\mu}p \int_{\Omega} K(x)w_{\mu}^{p-1}\psi_{\mu} w dx \geq \int_{\Omega} pK(x)w_{\mu}^{p-1}w\psi_{\mu} + p(\bar{\mu} - \mu) \int_{\Omega} K(x)w_{\mu}^{p-2}h\psi_{\mu} dx \geq \int_{\Omega} pK(x)w_{\mu}^{p-1}w\psi_{\mu} dx$$

which gives $\sigma_{\mu} > 1.$ \qed
Lemma 3.3. Suppose \((H_1)\), \(K(x) \geq 0\), \(K(x) \neq 0\) and \(K(x) \in L^1(\Omega)\). Then there exists a constant \(C > 0\) independent of \(\mu\) such that

\[
\|u_\mu - \mu h\|_{D_0^{1,2}(\Omega)} \leq C \quad \text{for all } \mu \in (0, \mu^*)
\]

where \(u_\mu\) is the minimal solution of \(1.1\) and \(h\) is the unique solution of \(2.7\).

Proof. Set \(v_\mu = u_\mu - \mu h\). From Lemma 2.4 we have

\[
\left\{ \begin{array}{l}
-\Delta v_\mu = K(x)(v_\mu + \mu h)^p, \\
v_\mu \in D_0^{1,2}(\Omega)
\end{array} \right. \quad (3.8)
\]

From Lemma 3.2 and \((3.8)\) we deduce

\[
\int_\Omega |\nabla v_\mu|^2 dx = \int_\Omega K(x)(v_\mu + \mu h)^p v_\mu dx \quad (3.9)
\]

\[
\int_\Omega |\nabla v_\mu|^2 dx \geq \sigma_\mu p \int_\Omega K(x)(v_\mu + \mu h)^{p-1} v_\mu^2 dx \quad (3.10)
\]

and hence

\[
\sigma_\mu p \int_\Omega K(x)(v_\mu + \mu h)^{p-1} v_\mu^2 dx \leq \int_\Omega K(x)(v_\mu + \mu h)^p v_\mu dx \leq \int_\Omega K(x)(v_\mu + \mu h)^{p-1} v_\mu^2 dx + \int_\Omega K(x)(v_\mu + \mu h)^{p-1} \mu h v_\mu dx.
\]

So, for any \(\epsilon > 0\),

\[
(p - 1) \int_\Omega K(x)(v_\mu + \mu h)^{p-1} v_\mu^2 dx \leq \int_\Omega K(x)\mu h(v_\mu + \mu h)^{p-1} v_\mu dx
\]

\[
\leq C \int_\Omega (K(x)v_\mu^p + K(x)v_\mu) dx
\]

\[
\leq C \left( \int_\Omega K(x) dx \right)^{\frac{p}{p+1}} \left( \int_\Omega K v_\mu^{p+1} dx \right)^{\frac{1}{p+1}}
\]

\[
+ C \left( \int_\Omega K(x) dx \right)^{\frac{p}{p+1}} \left( \int_\Omega (K(x)v_\mu^{p+1} dx) \right)^{\frac{1}{p+1}}
\]

\[
\leq C \epsilon \int_\Omega K(x) dx + \epsilon \int_\Omega K(x)v_\mu^{p+1} dx
\]

by Hölder’s inequality and Young’s inequality. Taking \(\epsilon > 0\) small enough we deduce

\[
\int_\Omega K(x)v_\mu^{p+1} dx \leq C \int_\Omega K(x) dx \leq C_1. \quad (3.11)
\]
From (3.9), (3.10) we also have
\[
\int_{\Omega} |\nabla v_{\mu}|^2 \, dx \leq \frac{1}{p} \int_{\Omega} |\nabla v_{\mu}|^2 \, dx + \int_{\Omega} K(x) \mu h (v_{\mu} + \mu h)^{p-1} v_{\mu} \, dx
\]
and hence
\[
\left(1 - \frac{1}{p}\right) \int_{\Omega} |\nabla v_{\mu}|^2 \, dx \leq C |\mu^*|^p \|h\|_{\infty}^p \int_{\Omega} K(x) v_{\mu} \, dx + C |\mu^*| \|h\|_{\infty} \int_{\Omega} K(x) v_{\mu}^p \, dx
\]
\[
\leq C \left(\int_{\Omega} K(x) \, dx\right)^{\frac{1}{p+1}} \left(\int_{\Omega} K(x) v_{\mu}^{p+1} \, dx\right)^{\frac{p}{p+1}}
\]
\[
+ C \left(\int_{\Omega} K(x) \, dx\right)^{\frac{1}{p+1}} \left(\int_{\Omega} K(x) v_{\mu}^{p+1} \, dx\right)^{\frac{p}{p+1}}
\]
\[
\leq C
\]
because of (3.11) and that \(K(x) \in L^1(\Omega)\).

\[\square\]

**Lemma 3.4.** Let \(h(x)\) be the solution of (2.7) and suppose \(H_1\), then for any \(\mu > 0\), there exists a unique solution \(u_\mu\) of (1.1)\(_\mu\) if \(K(x) \leq 0\). In addition, \(u_\mu \leq \mu h\) on \(\Omega\) and \(u_\mu\) is increasing in \(\mu\).

**Proof.** We remark that \(\mu h\) is a supersolution of (1.1)\(_\mu\) which satisfies
\[
\begin{cases}
-\Delta (\mu h) - K(x)(\mu h)^p \geq -\mu \Delta h = 0 & \text{in } \Omega \\
\mu h \big|_{\partial \Omega} = 0, \lim_{|x| \to \infty} \mu h(x) = \mu \\
\mu h > 0 & \text{in } \Omega
\end{cases}
\] (3.12)

Next, let \(\psi(x) = \int_{\Omega} G(x, y)|K(y)| \, dy\); from Lemma 2.2, \(\psi(x)\) is the positive solution of
\[
\begin{cases}
-\Delta v = |K(x)| \\
v \big|_{\partial \Omega} = 0, \ v(x) \to 0 & \text{as } |x| \to \infty
\end{cases}
\] (3.13)
we set \(\bar{u} = (\mu h - \lambda \psi)^{+}\) for some \(\lambda > 0\). We then have by standard results
\[
-\Delta \bar{u} \leq -\lambda |K(x)|_{\{u \geq 0\}} \leq K(x) \bar{u}^p \text{ on } \Omega
\]
if \(\lambda\) is chosen such that
\[
\bar{u}^p \leq (\mu h)^p \leq \lambda.
\]
where \( h(x) \) is the solution of (2.7). Thus \( \underline{u} \) is a nontrivial subsolution satisfies \( \underline{u} \leq \mu h \) and the existence part is complete.

The various uniqueness and comparison results are deduced from the following claim. Let \( v, w \in H^1_0(\Omega) \cap C_b(\Omega) \) satisfy

\[
-\Delta v + |K(x)|v^p \leq 0 \quad \text{in } \Omega \quad v \geq 0 \quad \text{on } \partial \Omega \quad \lim_{|x| \to \infty} v = \mu, \quad v \big|_{\partial \Omega} = 0,
\]

\[
-\Delta w + |K(x)|w^p \geq 0 \quad \text{in } \Omega \quad w \geq 0 \quad \text{on } \partial \Omega \quad \lim_{|x| \to \infty} w = \mu, \quad w \big|_{\partial \Omega} = 0,
\]

then \( v \leq w \) on \( \Omega \).

Indeed, for all \( \epsilon > 0 \), we may find \( R \) large enough such that

\[
v \leq (1 + \epsilon)w \equiv w_\epsilon \quad \text{for } |x| \geq R
\]

since we have on \( B_R \cap \Omega \)

\[
-\Delta (w_\epsilon - v) + p|K(x)|w_\epsilon^{p-1}(w_\epsilon - v) \\
\geq -\Delta (w_\epsilon - v) + |K(x)|(w_\epsilon^p - v^p) \\
= -\Delta w_\epsilon + |K(x)|w_\epsilon^p - (-\Delta v + |K(x)|v^p) \geq 0.
\]

Since the first eigenvalue of \( -\Delta + p|K(x)|w_\epsilon^{p-1} \) is positive (on \( H^1_0(\Omega \cap B_R) \)) we deduce \( w_\epsilon \geq v \) in \( \Omega \). Let \( \epsilon \to 0 \) we obtain our claim. Using the above claim we can easily deduce the uniqueness and that \( u_\mu \leq \mu h \) for all \( \mu > 0 \) and \( u_{\mu_1} \leq u_{\mu_2} \) if \( \mu_1 \leq \mu_2 \).

\[ \square \]

**Lemma 3.5.** Suppose \((H_1)\), if \( K(x) \) change sign, we can find a positive constant \( \mu^* \) such that problem \((1.1)_{\mu} \) possesses at least one solution.

**Proof.** Consider problem

\[
\begin{cases}
-\Delta v = K(x)(v + \mu h)^p, & v > 0 \quad \text{in } \Omega, \\
v \big|_{\partial \Omega} = 0, \lim_{|x| \to \infty} v(x) = 0.
\end{cases}
\]  

(3.14)

From Lemma 3.1, we can find a positive constant \( \mu^* \) such that problem

\[
\begin{cases}
-\Delta v = K^+(x)(v + \mu h)^p & \text{in } \Omega, \\
v \big|_{\partial \Omega} = 0, \lim_{|x| \to \infty} v = 0, \quad v > 0 \quad \text{in } \Omega
\end{cases}
\]
possess a minimal solution $\bar{v}$ for all $\mu \in (0, \mu^*)$. From Lemma 3.4, problem

$$\begin{cases}
-\Delta v = -K^- (x)(v + \mu h)^p & \text{in } \Omega, \\
v|_{\partial \Omega} = 0, \lim_{|x| \to \infty} v = 0, \ v < 0 & \text{in } \Omega.
\end{cases}$$

possesses a unique solution $v$ for all $\mu > 0$. Then $\bar{v}$ is a supersolution of (3.14)$_\mu$ and $v$ is a subsolution of (3.14)$_\mu$. Furthermore $v = \bar{v} - v_*$ satisfies

$$\begin{cases}
-\Delta v = K^+(x)(\bar{v} + \mu h)^p + K^-(x)(\underline{v} + \mu h)^p & \geq 0, \\
v|_{\partial \Omega} = 0, \lim_{|x| \to \infty} v = 0,
\end{cases}$$

maximum principle implies that $v > 0$. The existence of solution for (3.14)$_\mu$ with $K(x)$ change sign come from the method of super-subsolution. Suppose $v_\mu$ be the solution of (3.14)$_\mu$, then $u_\mu = v_\mu + \mu h$ is a solution of (1.1)$_\mu$ with $0 < \underline{v} + \mu h < u_\mu < \bar{v} + \mu h$.

\begin{proof}
From the above lemmas, we only have to prove that problem (1.1)$_{\mu^*}$ has a unique solution under the assumption (3.15). Denote the corresponding solution of (1.1)$_{\mu^*}$ by $u_{\mu^*}$.

Let $v_\mu = u_\mu - \mu h$. From assumption (3.15) and Lemma 3.3, we know $v_\mu \in D^{1,2}_0(\Omega)$ and

$$\|v_\mu\|_{D^{1,2}_0(\Omega)} \leq C < +\infty \text{ for all } \mu \in (0, \mu^*)$$

\end{proof}

**Theorem 3.6.** Suppose $(H_1)$. Let $h$ be the solution of (2.10), then

(i) If $K(x) \leq 0$, for any $\mu > 0$, there exists a unique solution $u_\mu$ of (1.1)$_\mu$. In addition, $u_\mu \leq \mu h$ on $\Omega$ and $u_\mu$ is increasing in $\mu$.

(ii) If $K(x) \geq 0$, $\exists \mu^* \in (0, \infty] \text{ and } \mu^* < +\infty \text{ if } K(x) \not\equiv 0$, such that for $\mu > \mu^*$ there does not exist a solution of (1.1)$_\mu$ and for $\mu \in (0, \mu^*)$, there exists a minimal solution $u_\mu$ of (1.1)$_\mu$. In addition, $u_\mu$ is increasing in $\mu$, $u_\mu \geq \mu h$ in $\Omega$. Finally, if

$$K(x) \in L^1(\Omega), \quad (3.15)$$

then as $\mu \to \mu^*$, $u_\mu$ increase to $u_{\mu^*}$ the minimal solution of (1.1)$_{\mu^*}$, and $u_{\mu^*}$ is unique.

(iii) If $K(x)$ change sign, we can find a $\mu^* \in (0, +\infty)$ such that problem (1.1)$_\mu$ possesses at least one solution for all $\mu \in (0, \mu^*)$.

**Proof.** From the above lemmas, we only have to prove that problem (1.1)$_{\mu^*}$ has a unique solution under the assumption (3.15). Denote the corresponding solution of (1.1)$_\mu$ by $u_\mu$. Let $v_\mu = u_\mu - \mu h$. From assumption (3.15) and Lemma 3.3, we know $v_\mu \in D^{1,2}_0(\Omega)$ and

$$\|v_\mu\|_{D^{1,2}_0(\Omega)} \leq C < +\infty \text{ for all } \mu \in (0, \mu^*)$$
where $C$ is a positive constant independent of $\mu$. We claim that

$$\int_{\Omega} v_{\mu}^q dx \leq C < \infty \quad (3.16)$$

for all $q \geq \frac{2N}{N-2}$, where $C$ is some positive constant independent of $N$. First of all, we consider $p \in (1, \frac{N+2}{N-2})$, the subcritical case. We adapt the argument due to Brezis and Kato [BK] to deduce the above claim. In fact, $v_{\mu}$ is a solution of

$$\begin{cases}
-\Delta v_{\mu} = K(x)(v_{\mu} + \mu h)^p \\
v_{\mu} \in D^{1,2}_0(\Omega) \quad v_{\mu} > 0 \quad \text{in } \Omega
\end{cases} \quad (3.17)$$

Let $i > 1$, multiplying $(3.17)_{\mu}$ by $v_i^\mu$ and integrating by parts we obtain

$$4i(1+i)^{-2} \int_{\Omega} |\nabla v_i^\mu|^{2(1+i)} dx = \int_{\Omega} K(x)(v_{\mu} + \mu h)^p v_i^\mu dx.$$
and (3.17)—(3.19) we have

\[
\left( \int_\Omega v_\mu^q dx \right) ^{\frac{N-2}{N}} = \left( \int_\Omega \left( v_\mu^\frac{1}{(1+i)} \right)^{\frac{2N}{N-2}} dx \right) ^{\frac{N-2}{N}} \\
\leq C \int_\Omega v_\mu^{p+i} dx + C \\
\leq C \epsilon \int_\Omega v_\mu^{i+\frac{N-2}{N}} dx + C \epsilon \int_\Omega v_\mu^{\frac{2N}{N-2}} dx + C \\
= C \epsilon \int_\Omega v_\mu^{\frac{(N-2)p}{N-2}} \cdot \left( \frac{4}{N-2} \right) dx + C \\
\leq C \epsilon \left( \int_\Omega v_\mu^q dx \right) ^{\frac{N-2}{N}} \left( \int_\Omega v_\mu^{\frac{2N}{N-2}} dx \right) ^{\frac{2}{N}} + C
\]

with \( q = \frac{N(1+i)}{N-2} \). From Lemma 3.3 and Sobolev inequality we deduce \( \{ v_\mu \} \) is bounded in \( L^q(\Omega) \) for large \( q > 1 \) if we choose \( \epsilon \) small enough.

Now, we are going to deal with the case when \( p = \frac{N+2}{N-2} \). Our method is a combination of ideas found in papers of Brezis and Kato [BK] and Egnell [E]. For \( j \geq 1 \), define \( \varphi_j(t) = t^j \), \( t \geq 0 \) and \( \psi_j(t) = \int_0^t |\varphi_j(s)|^2 ds = \frac{t^2}{2j-1} t^{2j-1} \). Let \( \mu \in (0, \mu^*) \) and \( v_\mu \) be the corresponding minimal solution of (3.17)\( \mu \). From Lemma 3.2 we have

\[
\int_\Omega \nabla v_\mu \nabla v dx \geq p \int_\Omega K(x)(v_\mu + \mu h)^{p-1} v dx
\]

(3.20)

for all \( v \in \mathcal{D}_0^{1,2}(\Omega) \). By Theorem 2.3, Lemma 2.4 and Remark 2.1 we know \( \varphi_j(v_\mu) \in \mathcal{D}_0^{1,2}(\Omega) \).

We may choose \( v = \varphi_j(v_\mu) \) in (3.20) to obtain

\[
\int_\Omega |\varphi'_j(v_\mu)|^2 |\nabla v|^2 dx \geq p \int_\Omega K(x)(v_\mu + \mu h)^{p-1} \varphi_j^2(v_\mu) dx.
\]

(3.21)

Since \( v_\mu \) is a solution of (3.17)\( \mu \) and \( \psi_j(v_\mu) \in \mathcal{D}_0^{1,2}(\Omega) \), we also have

\[
\int_\Omega \psi'_j(v_\mu) |\nabla v|^2 dx = \int_\Omega K(x)(v_\mu + \mu h)^p \psi_j(v_\mu) dx.
\]

(3.22)

From (3.22), (3.21) we obtain

\[
p \int_\Omega K(x)(v_\mu + \mu h)^{p-1} v_\mu^{2j} \leq \frac{j^2}{2j-1} \left[ \int_\Omega K(x)(v_\mu + \mu h)^{p-1} v_\mu^{2j} + \int_\Omega K(x)(v_\mu + \mu h)^{p-1} \mu h v_\mu^{2j-1} \right]
\]

(3.23)

since \( \frac{j^2}{2j-1} \geq 1 \) and is increasing in \( j \), we may choose \( j > 1 \) sufficiently close to 1 such that
\[ \frac{j^2}{2j-1} < p \text{ for } j \leq j_0. \] Set \( \alpha(j, p) = p - \frac{j^2}{2j-1} > 0. \) Then (3.23) gives

\[ \alpha(j, p) \int_{\Omega} K(x)v^{p+2j-1}dx \leq \alpha(j, p) \int_{\Omega} K(x)(v_\mu + \mu h)^{p-1}v_\mu^{2j}dx \]

\[ \leq \frac{j^2}{2j-1} \int_{\Omega} K(x)(v_\mu + \mu h)^{p-1}\mu h v_\mu^{2j-1}dx \]

\[ \leq C \frac{j^2}{2j-1} \left[ \int_{\Omega} K(x)v^{p+2j-2}_\mu \mu h dx + \int_{\Omega} K(x)(\mu h)^{p}v_\mu^{2j-1}dx \right] \]

\[ \leq C \left[ \int_{\Omega} K(x)v^{p+2j-2}_\mu dx + \int_{\Omega} K(x)v_\mu^{2j-1}dx \right]. \]

because \( \mu < \mu^*, K(x) \leq C. \) Since

\[ \int_{\Omega} K(x)v^{p+2j-2}_\mu dx \leq C \left[ \int_{\Omega} K(x)dx \right]^{\frac{1}{p+2j-1}} \left[ \int_{\Omega} K(x)v^{p+2j-1}_\mu dx \right]^{\frac{p+2j-2}{p+2j-1}} \]

\[ \leq C \int_{\Omega} K(x)dx + \frac{\delta}{2} \int_{\Omega} K(x)v^{p+2j-1}_\mu dx \]

for all \( \delta > 0 \) and similarly,

\[ \int_{\Omega} K(x)v^{2j-1}_\mu dx \leq C \int_{\Omega} K(x)dx + \frac{\delta}{2} \int_{\Omega} K(x)v^{p+2j-1}_\mu dx \]

for all \( \delta > 0, \) we can deduce

\[ (\alpha(j, p) - \delta) \int_{\Omega} K(x)v^{p+2j-1}_\mu dx \leq C_\delta \int_{\Omega} K(x)dx. \]

From the assumption of \( K(x) \in L^1(\Omega), \) we have

\[ \int_{\Omega} K(x)v^{p+2j-1}_\mu dx \leq C_\delta \quad (3.23)^* \]

for \( j \in (1, j_0] \) and \( C > 0 \) independent of \( \mu \in (0, \mu^*) \) if we take \( \delta \) small enough. This shows that (3.16) holds for all \( q \in \left[ \frac{2N}{N-2}, p + 2j_0 - 1 \right]. \) To establish (3.16) for all \( q \geq \frac{2N}{N-2} \) we use ideas in Brezis and Kato [BK]. Set \( q_0 = \frac{2N}{N-2}, \delta = p + 2j_0 - 1 - \frac{2N}{N-2} > 0. \) Multiplication of (3.17) by \( v_\mu^{q_0-1}, \) integration by parts and simple application of Hölder’s inequality and
Young’s inequality yield

\[(q_0 - 1)q_0^{-2} \left\| \nabla v_{\mu}^{\frac{q_0}{p}} \right\|^2 dx = \int_{\Omega} K(x)(v_{\mu} + \mu h)^p v_{\mu}^{q_0-1} dx \]

\[\leq C \int_{\Omega} K(x)v_{\mu}^{p+q_0-1} dx + C \int_{\Omega} K(x)v_{\mu}^{q_0-1} dx \]

\[\leq C \int_{\Omega} K(x)v_{\mu}^{p+q_0-1} dx \]

\[+ C \left( \int_{\Omega} K(x) dx \right)^{\frac{p}{p+q_0-1}} \left( \int_{\Omega} K(x)v_{\mu}^{p+q_0-1} dx \right)^{\frac{q_0-1}{p+q_0-1}} \]

\[\leq C \int_{\Omega} K(x)v_{\mu}^{p+q_0-1} dx + C \int_{\Omega} K(x) dx \]

which gives us

\[(q_0 - 1)q_0^{-2} \int \left\| \nabla v_{\mu}^{\frac{q_0}{p}} \right\|^2 dx \leq C \int_{\Omega} K(x)v_{\mu}^{p+q_0-1} dx + C \int_{\Omega} K(x) dx \] (3.24)

where \(C\) is a positive constant independent of \(\mu\).

For any given \(\epsilon > 0\) we can find a positive constant \(C_\epsilon\) such that

\[v_{\mu}^{p-1+q_0} \leq \epsilon v_{\mu}^{p-1+q_0} + \frac{2\delta}{N} + C_\epsilon v_{\mu}^{q_0}.\]

This can be easily verified by the fact that \(q_0 < p - 1 + q_0 < p - 1 + q_0 + \frac{2\delta}{N}\). Therefore, it follows from Hölder inequality, Sobolev’s inequality and (3.23)* with \(j = j_0\) that

\[\int_{\Omega} K(x)v_{\mu}^{p+q_0-1} dx \leq \epsilon \int_{\Omega} K(x)v_{\mu}^{p-1+q_0+\frac{2\delta}{N}} dx + C_\epsilon C \]

\[\leq \epsilon \left( \int_{\Omega} K(x) \left( v_{\mu}^{q_0} \right)^{\frac{p+1}{p}} dx \right)^{\frac{2\delta}{p+1}} \left( \int_{\Omega} K(x)v_{\mu}^{p+2j_0-1} dx \right)^{\frac{q_0}{p}} + C_\epsilon \]

\[\leq \epsilon C \int_{\Omega} (v_{\mu}^{q_0})^{p+1} dx + C_\epsilon C \]

\[\leq \epsilon C \int_{\Omega} \left| \nabla v_{\mu}^{\frac{q_0}{p}} \right|^2 dx + C_\epsilon C. \]

which gives us

\[\int_{\Omega} K(x)v_{\mu}^{p+q_0-1} dx \leq \epsilon C \int_{\Omega} \left| \nabla v_{\mu}^{\frac{q_0}{p}} \right|^2 dx + C_\epsilon C. \] (3.25)

It follows from (3.24) and (3.25), with \(\epsilon\) sufficiently small, that

\[\int_{\Omega} \left| \nabla v_{\mu}^{\frac{q_0}{p}} \right|^2 dx \leq C \] (3.26)
for some constant $C$, independent of $\mu$, and by Sobolev’s inequality we have
\[
\int_{\Omega} v_{\mu}^{q_0} \, dx \leq C.
\]
The desired inequality (3.16) then follows easily by iteration. Set $q_1 = \frac{q_0}{2}$ and $q_k = \frac{q_0}{2}^{k-1}$.

Denote $g_{\mu}(x) = K(x)(v_{\mu}(x) + \mu h(x))^p$. From the above proof we deduce $g_{\mu}(x) \in L^q(\Omega)$ for all $q \geq p + 1$ and
\[
\int_{\Omega} |g_{\mu}(x)|^q \, dx \leq C \int_{\Omega} K(x)^q v_{\mu}(x)^{pq} \, dx + C \int_{\Omega} K(x)^q (\mu h(x))^{pq} \, dx
\leq C |K(x)|_\infty^{q-1} \int_{\Omega} K(x)^q v_{\mu}^{pq} \, dx + C \mu^* |h(x)|_\infty^q \int_{\Omega} K(x)^q \, dx
\leq C
\]
for all $\mu \in (0, \mu^*)$.

We employ a classical a priori estimate to obtain
\[
\|v_{\mu}\|_{C^1,0(B_R \cap \Omega)} \leq C_R (\|v_{\mu}\|_{p+1,B_2R(\Omega)} + \|g_{\mu}\|_{q,B_2R(\Omega)} )
\]
for solution of $-\Delta v = g_{\mu}(x)$, where $B_R(x)$ is a ball of radius $R$ and centre $x$, and $C_R$ is a constant independent of $\mu$ and $x$. Hölder estimates in $B_R \cap \Omega$ then shows that
\[
\|v_{\mu}\|_{C^{1,\alpha}(B_R \cap \Omega)} \leq C_R
\]
for some constant $C_R$, independent of $\mu$. A simple diagonalization argument and the Ascoli-Arzela theorem may be employed to show that for a subsequence $\mu_n \to \mu^*$, $v_{\mu_n}, |\nabla v_{\mu_n}|$ converge uniformly on each compact subset of $\Omega$, to a function $v_{\mu^*} \in D_0^{1,2}(\Omega)$. It follows that
\[
\int_{\Omega} \nabla v \cdot \nabla v_{\mu^*} \, dx = \int_{\Omega} K(x)(v_{\mu^*} + \mu^* h)^p v \, dx
\]
for all $v \in C_0^\infty(\Omega)$ and therefore $v_{\mu^*}$ is a nonnegative weak solution of (3.17)$_\mu$. Thus $u_{\mu^*} = v_{\mu^*} + \mu^* h$ is a solution of (1.1)$_{\mu^*}$.

Finally, we prove that $u_{\mu^*}$ is unique. In fact, from the definition we can easily deduce that $\sigma_{\mu^*} = 1$ by applying the implicit function theorem to the function $F : D_0^{1,2}(\Omega) \to D_0^{1,2}(\Omega)$ with
\[
F(u) = -\Delta u - K(x)(u + \mu h)^p \quad u \in D_0^{1,2}(\Omega).
\]
If there exists another solution $\bar{u}_{\mu^*} \geq u_{\mu^*}$ for problem $(1.1)_{\mu^*}$, set $\bar{v}_{\mu^*} = \bar{u}_{\mu^*} - \mu^* h$, $v_{\mu^*} = u_{\mu^*} - \mu^* h$. We have from (3.17)$_\mu$

\[-\Delta (\bar{v}_{\mu^*} - v_{\mu^*}) = K(x)[(\bar{v}_{\mu^*} + \mu^* h)^p - (v_{\mu^*} + \mu^* h)^p]
= K(x)[p(v_{\mu^*} + \mu^* h)^{p-1}(\bar{v}_{\mu^*} - v_{\mu^*})
+ p(p-1)(v_{\mu^*} + \theta(\bar{v}_{\mu^*} - v_{\mu^*}) + \mu^* h)^{p-2}(\bar{v}_{\mu^*} - v_{\mu^*})]\]

for some $\theta(x) \in [0,1]$. From Lemma 3.2 and the above equality, we deduce

\[\sigma(\mu^*) \int_\Omega p(v_{\mu^*} + \mu^* h)^{p-1}(\bar{v}_{\mu^*} - v_{\mu^*}) \psi_{\mu^*} dx = \int_\Omega \nabla \psi_{\mu^*} \nabla (\bar{v}_{\mu^*} - v_{\mu^*}) dx\]
\[= \int_\Omega p(v_{\mu^*} + \mu^* h)^{p-1}(\bar{v}_{\mu^*} - v_{\mu^*}) \psi_{\mu^*} dx + \int_\Omega p(p-1)(v_{\mu^*} + \theta(\bar{v}_{\mu^*} - v_{\mu^*}) + \mu^* h)^{p-2}(\bar{v}_{\mu^*} - v_{\mu^*})^2 \psi_{\mu^*} dx\]

i.e. \[(\sigma(\mu^*) - 1) \int_\Omega p(v_{\mu^*} + \mu^* h)^{p-1}(\bar{v}_{\mu^*} - v_{\mu^*}) \psi_{\mu^*} dx = \int_\Omega p(p-1)(v_{\mu^*} + \theta(\bar{v}_{\mu^*} - v_{\mu^*}) + \mu^* h)^{p-2}(\bar{v}_{\mu^*} - v_{\mu^*})^2 \psi_{\mu^*} dx\]

we can obtain that $\bar{v}_{\mu^*} \equiv v_{\mu^*}$ from $\sigma(\mu^*) = 1$. \[\square\]

4 The existence of second solution

For $\mu \in (0, \mu^*)$, let $u_{\mu}$ be the first solution of $(1.1)_{\mu}$ and consider the problem

\[
\begin{align*}
-\Delta v &= K(x)((v + u_{\mu})^p - u_{\mu}^p) \quad \text{in } \Omega \\
v &\in D_0^1(\Omega) , \quad v > 0 \quad \text{in } \Omega.
\end{align*}
\]

(4.1)$_\mu$

It is clear that $U_{\mu} = v_{\mu} + u_{\mu}$ is a solution of $(1.1)_{\mu}$ if $v_{\mu}$ is a solution of $(4.1)_{\mu}$. Consider the energy functional $J_{\mu}$ defined by

\[J_{\mu}(v) = \int_{\Omega} \frac{1}{2} |\nabla v|^2 - K(x) \left[ \frac{1}{p+1}(u_{\mu} + v^+)^{p+1} - \frac{1}{p+1}u_{\mu}^{p+1} - u_{\mu}^p v^+ \right] dx .\]

Standard procedure from the calculus of variations shows that $J_{\mu}$ is well defined in $D_0^1(\Omega)$ with continuous Frechet derivative given by

\[J'_{\mu}(v)\phi = \int_{\Omega} [\nabla v \nabla \phi - K(x)((u_{\mu} + v^+)^p - u_{\mu}^p)] \phi dx \quad \phi \in D_0^{1,2}(\Omega)\]
A critical point \( v \) of \( J_\mu \) is a weak solution of the equation
\[
-\Delta v = K(x)[(u_\mu + v^+)^p - u_\mu^p]\quad v \in \mathcal{D}_0^1(\Omega)
\]
and if \( v > 0 \) in \( \mathbb{R}^N \), then \( v \) is a solution of (4.1).µ.

The following Lemma comes from the fact that
\[
\lim_{s \to 0} \frac{(u_\mu + s)^p - u_\mu^p - pu_\mu^{p-1}s}{s} = 0
\]
and
\[
\lim_{s \to \infty} \frac{(u_\mu + s)^p - u_\mu^p - pu_\mu^{p-1}s}{sp^p} = 1.
\]

**Lemma 4.1.** For any \( \epsilon > 0 \), there exist a \( C_\epsilon > 0 \) such that
\[
(u_\mu + s)^p - u_\mu^p - pu_\mu^{p-1}s \leq \epsilon u_\mu^{p-1}s + C_\epsilon s^p
\]
for all \( s \geq 0 \).

**Lemma 4.2.** Suppose \((H_1)\) with \( \ell = -\frac{N+2}{2} - \epsilon \). There exists two constant \( \alpha > 0, \rho > 0 \) such that
\[
J_\mu(v) \geq \alpha > 0, \quad \text{for } v \in \mathcal{D}_0^1(\Omega), \quad \|v\| = \rho .
\]

**Proof.** Lemma 4.1 implies that
\[
J_\mu(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \frac{p}{2} \int_\Omega K(x)u_\mu^{p-1}(v^+)^2 dx
\]
\[
- \int_\Omega \int_0^{v^+} K(x)[(u_\mu + s)^p - u_\mu^p - pu_\mu^{p-1}s]ds dx
\]
\[
\geq \frac{1}{2} \int_\Omega |\nabla v|^2 - pK(x)u_\mu^{p-1}(v^+)^2 dx
\]
\[
- \int_\Omega K(x) \left( \frac{\epsilon}{2} u_\mu^{p-1}(v^+)^2 + C_\epsilon \frac{(v^+)^{p+1}}{p+1} \right) dx .
\]
Furthermore, from the definition of \( \sigma_\mu \) in Lemma 3.2, we have
\[
\int_\Omega |\nabla v|^2 dx \geq \sigma_\mu p \int_\Omega K(x)u_\mu^{p-1}(v^+)^2 dx
\]
and, therefore, by \( \sigma_\mu > 1 \) we obtain by choosing \( \epsilon \) small enough
\[
J_\mu(v) \geq \frac{1}{2\sigma_\mu}(\sigma_\mu - 1 - \epsilon) \int_\Omega |\nabla v|^2 dx - \frac{C_\epsilon}{p+1} \int_\Omega K(x)v^{p+1} dx
\]
\[
\geq \frac{1}{4\sigma_\mu}(\sigma_\mu - 1) \int_\Omega |\nabla v|^2 dx - C \left[ \int_\Omega |\nabla v|^2 dx \right]^{p+1}
\]
\[
= \frac{1}{4\sigma_\mu}(\sigma_\mu - 1)\|v\|^2 - C\|v\|^{p+1}
\]
and the conclusion in Lemma 4.2 follows.
Lemma 4.3. Suppose $(H_1)$ with $\ell = -\frac{N+2}{2} - \epsilon$. Then there exist $0 < \psi_0 \in D^1_0(\Omega)$ and $R_0 > 0$ such that

$$J_\mu(R\psi_0) < 0$$

for $R \geq R_0$.

Proof. Let $h(x, s) = K(x)((u_\mu + s)^p - u_\mu^p - s^p)$, since $u_\mu(x)$ is bounded in $\Omega$, it is easy to check that

$$\lim_{s \to 0} \frac{h(x, s)}{s} \leq M$$
$$\lim_{s \to \infty} \frac{h(x, s)}{s^p} = 0$$

uniformly in $x \in \Omega$, where $M > 0$ is some constant independent of $x$. Therefore, for any $\epsilon > 0$, there is a constant $C_\epsilon > 0$ such that

$$h(x, s) \leq \epsilon s^p + C_\epsilon s .$$

Now, choose a nonzero function $\psi_0 \in C^\infty_0(\Omega)$ such that $\psi_0 \geq 0$ and $K(x) \geq k_0 > 0$ on the support of $\psi_0$. Then

$$J_\mu(R\psi_0) \leq \frac{1}{2} R^2 \|\psi_0\|^2 - \frac{R^{p+1}}{p + 1} \int_\Omega K(x)\psi_0^{p+1} dx + C_\epsilon R^2 \int_\Omega K\psi_0^2 dx + \epsilon R^{p+1} \int_\Omega K\psi_0^{p+1} dx .$$

It is then clear from the choice of $\psi_0$, that for $\epsilon$ sufficiently small there is $R_0 > 0$ such that

$$J_\mu(R\psi_0) < 0$$

for all $R \geq R_0$.

This completes the proof of Lemma 4.3, with $R_0$ and $\psi_0$ as above. \hfill \square

In order to use mountain pass Lemma [BN] to obtain the solution of $(4.1)_\mu$, we suppose moreover $(H_2)$.

Set

$$\Gamma = \{ \gamma \in C([0, 1], \ D^{1,2}_0(\Omega)), \ \gamma(0) = 0, \ \gamma(1) = R_0\psi_0 \},$$

where $\psi_0$ is given by Lemma 4.2. We exploit the fact that the critical equation

$$-\Delta u = u^{\frac{N+2}{N-2}} \quad \text{in } \mathbb{R}^N$$
has the positive radial solution
\[ u_\epsilon(x) = k \left[ \frac{\epsilon}{\epsilon^2 + |x - x_0|^2} \right]^{N-2} \]
with \( k = (N(N - 2))^{\frac{N-2}{4}} \) for any \( \epsilon > 0, \ x \in \mathbb{R}^N \). Furthermore,
\[ \int_{\mathbb{R}^N} |\nabla u_\epsilon|^2 dx = \int_{\mathbb{R}^N} u_\epsilon^{p+1} dx = S^{N/2}, \]
and for some positive constant \( c \) depending only on \( N \) \( cu_\epsilon(x) \) attains the infimum for the variational problem
\[ S = \inf \left\{ ||u||^2 \mid \int_{\mathbb{R}^N} u^{p+1} dx = 1 \ u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N) \right\}. \]

Let \( R > 0 \) be small enough that \( B_{2R}(x_0) \in V \). Let \( \psi \) be a piecewise smooth function with support in \( B_{2R} \) such that \( \psi(x) \equiv 1 \) in \( B_R(x_0) \), \( 0 \leq \psi(x) \leq 1 \) in \( B_{2R}(x_0) \) and \( |\nabla \psi(x)| \leq \frac{1}{R} \).

Define
\[ w_\epsilon(x) = \psi(x)u_\epsilon(x) \]
and
\[ v_\epsilon(x) = w_\epsilon(x) \left[ \int_\Omega K(x)w_\epsilon^{p+1} dx \right]^{\frac{1}{p+1}}. \]

The proof of the following Lemma follows the same lines as in [BK].

**Lemma 4.4.** If assumptions (H1) - (H2) holds and \( p = \frac{N+2}{N-2} \), then there exist some positive constant \( \epsilon > 0 \) and \( t_0 > 0 \) such that
\[ J_\mu(t_0u_\epsilon) < 0 \]
and
\[ 0 < \sup_{t \geq 0} J_\mu(tu_\epsilon) < \frac{1}{N} S^{N/2} (\|K\|_{L^\infty})^{\frac{2-N}{2}}. \]

**Proof.** Since \( \frac{\partial u_\epsilon}{\partial \gamma} \leq 0 \), we have
\[ \int_{B_R} |\nabla w_\epsilon|^2 dx = \int_{B_R} |\nabla u_\epsilon|^2 dx \leq \int_{B_R} u_\epsilon^{p+1} dx \]
and by the assumption (H\textsubscript{2}) we also have
\[ K(x_0) \int_{B_R} u_{\epsilon}^{p+1} dx \leq \int_{B_R} K(x) u_{\epsilon}^{p+1} dx + 0(\epsilon^2). \]

Simple calculations also show that
\[ \int_{\mathbb{R}^N \setminus B_R} u_{\epsilon}^{p+1} dx = 0(\epsilon^N) \]
\[ A_\epsilon \equiv \int_{\mathbb{R}^N \setminus B_R} |\nabla w_{\epsilon}|^2 dx = 0(\epsilon^{N-2}) \]
as \( \epsilon \to 0 \) and
\[ S = \left[ \int_{\mathbb{R}^N} u_{\epsilon}^{p+1} dx \right]^{\frac{2}{p+1}}. \]

Therefore, we have
\[ \int_{\mathbb{R}^N} |\nabla w_{\epsilon}|^2 dx = \int_{B_R} |\nabla w_{\epsilon}|^2 dx + A_\epsilon \]
\[ \leq \int_{B_R} u_{\epsilon}^{p+1} dx + A_\epsilon \]
\[ \leq S \left[ \int_{B_R} u_{\epsilon}^{p+1} dx \right]^{\frac{2}{p+1}} + A_\epsilon \]
\[ \leq S \|K\|_\infty^{-\frac{2}{p+1}} \left[ \int_{B_R} K(x) u_{\epsilon}^{p+1} dx \right] + 0(\epsilon^2) + 0(\epsilon^{N-2}). \]

Set \( V_\epsilon \equiv \int_{\mathbb{R}^N} |\nabla v_{\epsilon}|^2 dx \), since for small \( \epsilon > 0 \), say \( \epsilon \leq \epsilon_0 \), it is easy to see that
\[ \int_{B_R} K(x) u_{\epsilon}^{p+1} dx \geq C_{\epsilon_0} \]
for some positive constant \( C_{\epsilon_0} \), the definition of \( V_\epsilon \) and the last two inequalities imply that
\[ V_\epsilon \leq S(\|K\|_\infty^{-\frac{2}{p+1}} + 0(\epsilon^2) + 0(\epsilon^{N-2}). \]

We consider now \( J_\mu(v) \)
\[ J_\mu(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \frac{1}{p+1} \int_{\Omega} K(x)|(u_\mu + v^+)^{p+1} - u_\mu^{p+1}| dx + \int_{\Omega} K(x)u_\mu^p v^+ dx \]
\[ = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} K(x) \int_0^{v^+} ((u_\mu + s)^p - u_\mu^p) ds dx \]
\[ = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \frac{1}{p+1} \int_{\Omega} K(x)(v^+)^{p+1} dx \]
\[ - \int_{\Omega} K(x) \int_0^{v^+} [(u_\mu + s)^p - u_\mu^p - s^p] ds dx. \]
Set $F(x, v) = K(x) \int_0^{v^+} ((u_\mu + s)^p - u_\mu^p - s^p)ds$, then

$$J_\mu(tv_\epsilon) = \frac{1}{2} t^2 V_\epsilon - \frac{1}{p+1} t^{p+1} - \int F(x, tv_\epsilon)dx .$$

Clearly, $\lim_{t \to \infty} J_\mu(tv_\epsilon) = -\infty$ for all $\epsilon > 0$, hence $\sup_{t \geq 0} J_\mu(tv_\epsilon)$ is attained by some $t_\epsilon \geq 0$, we may assume $t_\epsilon > 0$ for $\epsilon > 0$, otherwise there would be nothing to prove.

It follows from $\frac{d}{dt} J_\mu(tv_\epsilon)|_{t=t_\epsilon} = 0$ and the monotonicity of $F$ in $v$ that

$$t_\epsilon \leq V_\epsilon^{\frac{1}{p+1}} \leq C_0 \text{ for all } \epsilon > 0 ,$$

where $C_0$ is some positive constant independent of $\epsilon$. By the monotonicity property of $\frac{1}{2} t^2 V_\epsilon - \frac{1}{p+1} t^{p+1}$ on the interval $(0, V_\epsilon^{\frac{1}{p+1}}]$ we then have

$$\sup_{t \geq 0} J_\mu(tv_\epsilon) = J_\mu(t_\epsilon v_\epsilon) \leq \frac{1}{N} V_\epsilon^{\frac{1}{p+1}} - \int_{B_{2R}} F(x, t_\epsilon v_\epsilon)dx .$$

The estimate on $V_\epsilon$ and the above inequality imply that

$$\sup_{t \geq 0} J_\mu(tv_\epsilon) \leq \frac{1}{N} S_{\frac{N}{p+1}}(\|k\|_\infty) - \int_{B_{2R}} F(x, tv_\epsilon)dx + 0(\epsilon^L) ,$$

where $L = \min(N - 2, 2)$. The conclusion will follows if we can show that

$$\lim_{\epsilon \to 0} \epsilon^{-L} \int_{B_{2R}} F(x, t_\epsilon v_\epsilon)dx = +\infty .$$

First we claim that

$$\lim_{\epsilon \to 0} t_\epsilon > 0 .$$

Indeed, by $\frac{d}{dt} J_\mu(tv_\epsilon)|_{t=t_\epsilon} = 0$ we have

$$V_\epsilon - t_\epsilon^{p+1} - t_\epsilon^{-1} \int_{\Omega} K(x)[(u_\mu + t_\epsilon v_\epsilon)^p - u_\mu^p - t_\epsilon^p v_\epsilon^p]v_\epsilon dx = 0 .$$

We show that

$$\lim_{\epsilon \to 0} t_\epsilon^{-1} \int_{\Omega} K(x)[(u_\mu + t_\epsilon v_\epsilon)^p - u_\mu^p - t_\epsilon^p v_\epsilon^p]v_\epsilon dx = 0 .$$

This will follows by the same procedure as in [BK] (p465–466) by observing first that for all $\delta > 0$, $\exists C_\delta > 0$ such that

$$|f(x, u)| \equiv |(u_\mu + u)^p - u_\mu^p - u^p| \leq \delta u^p + C_\delta u ,$$
for all \( u > 0 \). This follows easily from the boundedness of \( u_\mu \). Indeed for \( u \geq \frac{1}{\delta} \), we have

\[
|f(x, u)| = u^p \int_0^{u_\mu} ((s + 1)^{p-1} - s^{p-1}) ds \leq C u^p
\]

for some constant \( C \). For \( u \leq \frac{1}{\delta} \) we have

\[
|f(x, u)| \leq \frac{(u_\mu + u)^p - u_\mu^p}{u} u + u^p \\
\leq p(u_\mu + u)^{p-1} + \left( \frac{1}{\delta} \right)^{p-1} \\
\leq C u.
\]

It then follows that a positive constant \( C \), independent of \( \epsilon \), exists such that

\[
\sup_{t \geq 0} J_\mu(tv_\epsilon) \leq \frac{1}{N} \|K\|_\infty \left( \sum_{j=1}^{2N} \left( 2^{2N} - 1 \right) \right) - \int_{B_2R} F(x, Cv_\epsilon) dx + 0(\epsilon L)
\]

for sufficiently small \( \epsilon > 0 \). A change of variables yields

\[
\lim_{\epsilon \to 0^+} \epsilon^{-L} \int_{B_2R} F(x, Cv_\epsilon) dx = +\infty
\]

as in [BN].

\[\Box\]

**Lemma 4.5.** Assume \( H_2 \) and \( H_1 \). Suppose moreover \( 0 \not\equiv K(x) \geq 0 \) and \( K(x) \in L^1(\Omega) \). Then problem (4.1)_\mu has at least two solution for each \( \mu \in (0, \mu^*) \) if \( p = \frac{N+2}{N-2} \).

**Proof.** The conditions for the mountain pass Lemma [BN] are satisfied by Lemma 4.2, 4.3. Hence there is a sequence \( \{v_n\} \subset \mathcal{D}_0^1(\Omega) \) such that \( J_\mu(v_n) \to c \) and \( J'_\mu(v_n) \to 0 \) in \( \mathcal{D}_0^1(\Omega) \) as \( n \to \infty \), where

\[
c = \inf_{\nu \in \Gamma} \sup_{u \in \nu} J_\mu(u).
\]

Thus

\[
J_\mu(v_n) = \frac{1}{2} \int_\Omega |\nabla v_n|^2 dx - \int_\Omega \left[ \frac{1}{p+1} K(x)(u_\mu + v_n^+)^{p+1} - \frac{1}{p+1} w_\mu^{p+1} - u_\mu^p v_n^+ \right] dx = c + o(1),
\]

and

\[
J'_\mu(v_n)\psi = \int_\Omega \nabla v_n \nabla \psi dx - \int_\Omega K((u_\mu + v_n^+)^p - u_\mu^p) \psi dx = o(1) \|
\psi\|
\]
as \( n \to \infty \) and \( \psi \in D_0^1(\Omega) \). Choose \( \frac{1}{p+1} < \theta < \frac{1}{2} \) and \( \psi = v_n \). It follows from (4.2), (4.3) that
\[
c + o(1) \geq \frac{1}{2} \int_\Omega |\nabla v_n|^2 \, dx - \frac{1}{p+1} \int_\Omega K(u_\mu + v_n^+)^p \, dx
\]
\[
= \left( \frac{1}{2} - \theta \right) ||v_n||^2 + \theta(||v_n||^2 - \int_\Omega K[(u_\mu + v_n^+)^p - u_\mu^p]v_n \, dx)
\]
\[
+ \left( \theta - \frac{1}{p+1} \right) \int_\Omega K(x)(u_\mu + v_n^+)^p v_n^+ \, dx - \theta \int_\Omega K u_\mu^p v_n^+ \, dx
\]
\[
= \left( \frac{1}{2} - \theta \right) ||v_n||^2 + o(1)||v_n||
\]
\[
+ \left( \theta - \frac{1}{p+1} \right) \int_\Omega K(x)(v_n^+ - \tau u_\mu)(u_\mu + v_n^+)^p \, dx
\]
\[
- \theta \int_\Omega K(x) u_\mu^p v_n^+ \, dx ,
\]
where \( \tau = (p+1)^{-1}(\theta - (p+1)^{-1})^{-1} \). Notice that we have used the obvious equality
\[
\int_\Omega K(x)((u_n + v_n^+)^p - u_\mu^p)v_n \, dx = \int_\Omega K[(u_\mu + v_n^+)^p - u_\mu^p]v_n^+ \, dx .
\]
Using Hölder’s and Sobolev’s inequality we have
\[
\int_\Omega K u_\mu^p v_n^+ \, dx \leq ||u_\mu||_\infty^p \int_\Omega K(x)v_n^+ \, dx
\]
\[
\leq C \left( \int_\Omega K(x)^2 \, dx \right)^{1/2} \left( \int v_n^{+2} \, dx \right)^{1/2}
\]
\[
\leq C_1 ||v_n^+||
\]
for some constant \( C > 0 \).

Because \( g(x) = (s - \tau u_\mu)(u_n + s)^p \) gets its minimum at
\[
s = \frac{p\tau - 1}{1+p}
\]
we have
\[
c + o(1) \geq \left( \frac{1}{2} - \theta \right) ||v_n||^2 + o(1)||v_n||
\]
\[
- \frac{2(p(1+\tau))(\theta(p+1) - 1)}{(p+1)^{p+2}} \int_\Omega K(x) u_\mu^{p+1} \, dx
\]
\[
- \theta C_1 ||v_n||
\]
since $\|u_\mu\|_{L^\infty}$ is bounded, $K(x) \in L^1(\Omega)$. From the above inequality we can deduce \{v_n\} is bounded in $D_0^1(\Omega)$. Standard embedding theorem then show that \{v_n\} has a subsequence, still denoted by \{v_n\} for which

$$v_n \rightharpoonup v \text{ weakly in } D_0^1(\Omega)$$

$$v_n \to v \text{ a.e. in } \Omega$$

$$v_n \rightharpoonup v \text{ weakly in } L^{p+1}(\Omega)$$

It follows from (4.2) and (4.3) that $v$ is a weak solution of

$$-\Delta v = K(x)[(u_\mu + v^+)^p - u_\mu^p] \quad v \in D_0^1(\Omega)$$

Furthermore, (4.3) with $\psi = v^-$ implies that $\int_\Omega |\nabla v^-|^2 \, dx = 0$ and therefore $\int_\Omega |v^-|^{p+1} \, dx = 0$, by Sobolev embedding. This shows that $v \geq 0$ a.e. in $\Omega$, we show next that $v \not\equiv 0$.

Consider the sequence \{w_n\}, $w_n = v_n - v$, for a subsequence of \{w_n\}, denoted the same way, we define

$$\ell = \lim_{n \to \infty} \|w_n\|^2.$$  

If $\ell = 0$, the continuity of $J_\mu$ on $D_0^1(\Omega)$ implies that

$$0 < \alpha \leq c = \lim_{n \to \infty} J_\mu(v_n) = J_\mu(v) \quad \text{and hence } v \not\equiv 0.$$  

If $\ell > 0$, we proceed as follows. Using (4.3) with $\psi = v_n$, the boundedness of $\|v_n\|$, the weak convergence of $v_n$ to $v$ in $L^{p+1}(\Omega)$ and the fact that $u_\mu \in L^\infty(\Omega)$, $K(x) \in L^{\frac{n+1}{p}}$ we obtain

$$\int_\Omega K(x)(u_\mu + v_n)^p u_\mu \, dx \to \int_\Omega K(x)(u_\mu + v)^p u_\mu \, dx$$

$$\int_\Omega K(x)u_\mu^p v_n \, dx \to \int_\Omega ku_\mu^p v \, dx.$$  

We have

$$\int_\Omega |\nabla v_n|^2 \, dx - \int_\Omega K(x)(u_\mu + v_n^+)^{p+1} \, dx + \int_\Omega K(u_\mu + v)^p u_\mu \, dx + \int_\Omega K(x)u_\mu^p v \, dx = o(1). \quad (4.4)$$

Using a lemma of Bresis and Lieb [BL] and (4.4) we obtain

$$\int_\Omega |\nabla w_n|^2 \, dx + \int_\Omega |\nabla v|^2 \, dx - \int_\Omega K(x)(v_n^+ - v)^{p+1} \, dx$$

$$= \int_\Omega K(x)((u_\mu + v)^p - u_\mu^p) v \, dx + o(1).$$
Since \( v \) is a solution of problem (4.1), we have

\[
\int_{\Omega} |\nabla w_{n}|^2 dx = \int_{\Omega} K(x)(v^+ - v)^{p+1} dx + o(1). \tag{4.5}
\]

Using (4.3) and Bresis-Lieb Lemma [BL] we also have

\[
o(1) + c = \frac{1}{2} \int_{\Omega} |\nabla w_{n}|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \frac{1}{p+1} \int_{\Omega} K(x)w_{n}^{p+1} dx \\
- \frac{1}{p+1} \int_{\Omega} K(x)(u_{\mu} + v)^{p+1} dx + \frac{1}{p+1} \int_{\Omega} K(x)u_{\mu}^{p+1} dx + \int_{\Omega} K(x)u_{\mu}^p v dx \\
= \frac{1}{2} \int_{\Omega} |\nabla w_{n}|^2 dx - \frac{1}{p+1} \int_{\Omega} K(v_{n}^+ - v)^{p+1} dx + J_{\mu}(v). 
\]

which gives us

\[
o(1) + c = \frac{1}{2} \int_{\Omega} |\nabla w_{n}|^2 dx - \frac{1}{p+1} \int_{\Omega} K(v_{n}^+ - v)^{p+1} dx + J_{\mu}(v). \tag{4.6}
\]

It follows from (4.5) and (4.6) that

\[
c = \frac{1}{N} \ell + J_{\mu}(v). 
\]

We also have by Sobolev’s inequality and (4.5) that

\[
\int_{\Omega} |\nabla w_{n}|^2 dx \geq S \left( \int_{\Omega} |w_{n}|^{p+1} dx \right)^{\frac{2}{p+1}} \\
\geq S \left( \int_{\Omega} |v_{n}^+ - v|^{p+1} dx \right)^{\frac{2}{p+1}} \\
\geq \left( \frac{1}{\sup_{\Omega} K(x)} \right)^{\frac{2}{p+1}} S \left( \int_{\Omega} K(x)(|v_{n}^+ - v|^{p+1}) dx \right)^{\frac{2}{p+1}} \\
= \left( \frac{1}{\sup_{\Omega} K} \right)^{\frac{2}{p+1}} S(\|w_{n}\|^2 + o(1))^{\frac{2}{p+1}}
\]

which gives in the limit, as \( n \to \infty \), the inequality

\[
\ell \geq \left( \sup_{\Omega} K(x) \right)^{-\frac{2}{p+1}} S \ell^{\frac{2}{p+1}} \tag{4.8}
\]

since \( \ell > 0 \), (4.7) and (4.8) give

\[
c \geq \frac{1}{N} \left( \sup_{\Omega} K(x) \right)^{\frac{2}{p+1}} S^\frac{2}{p+1} + J_{\mu}(v) \tag{4.9}
\]

which implies from Lemma 4.4 that \( J_{\mu}(v) < 0 \), thus \( v \not\equiv 0 \). \( \square \)

From the above lemmas we conclude the theorem 1.2.
References


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