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Eigenfunction and harmonic function estimates in
domains with horns and cusps

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1 Introduction

An interesting example in the paper of Davies and Simon [5] was that of a horn-shaped domain in $\mathbb{R}^d$. By horn-shaped we mean domains of the form $D = \{(x, y) : x > 0, \|y\| < f(x)\}$ with $f : [0, \infty) \to (0, \infty)$ a function tending to zero as $x$ tends to infinity. Davies and Simon [5] established sufficiently sharp estimates on the first Dirichlet eigenfunction of $\Delta_d$ ($d$-dimensional Laplacian) on $D$ to determine when the Dirichlet heat semigroup on $D$ is intrinsically ultracontractive. This last property is important as one can provide bounds in that case for all the eigenfunctions, the heat kernel, and Green function in terms of the first eigenfunction. Thus, if one gets precise bounds on the first eigenfunction and the domain is intrinsically ultracontractive, then one has precise estimates on other important analytic quantities associated to the domain. Several works have appeared providing such bounds, Bañuelos [1],[2], Bañuelos and van den Berg [3], Bañuelos and Davis [4], Lindemann, Pang and Zhao [8].

In this paper we shall obtain pointwise bounds for positive harmonic functions vanishing
on the lateral side of horn-shaped domains in $\mathbb{R}^d$ as well as for Dirichlet eigenfunctions in these domains. Our most general horn-shaped domain is as follows. Let $R : [0, \infty) \mapsto U(d)$, the unitary group, $S : [0, \infty) \mapsto \mathbb{R}^{d-1}$, $f : [0, \infty) \mapsto (0, \infty)$ and suppose all are continuous (more conditions will be added in due course) and take $\Omega$ to be a bounded domain in $\mathbb{R}^{d-1}$. Then our domains are of the form

$$D = \{ (x, y) : x > 0, R(x)(y - S(x)) / f(x) \in \Omega \}.$$

Throughout this paper we will always assume that

$$\lim_{x \to \infty} f(x) = 0 \text{ and } \partial \Omega \in C^{2, \alpha},$$

for some $\alpha > 0$. Suppose $\tilde{w}_1$ is the first Dirichlet eigenfunction of the Laplacian for $\Omega$ and $\tilde{\lambda}_1$ is its corresponding eigenvalue. What we establish is that whenever $u$ is positive and harmonic in $D$, vanishing at infinity and on $\partial D \cap \{ (x, y) : x > 0 \}$ then $u(x, y)$ is bounded from above and below by constant multiples of $\exp \left\{ -\sqrt{\tilde{\lambda}_1} - \epsilon \int_0^x \frac{dr}{f(r)} \right\} \tilde{w}_1 \left( \frac{R(x)(y - S(x))}{f(x)} \right)$ and $\exp \left\{ -\sqrt{\tilde{\lambda}_1} + \epsilon \int_0^x \frac{dr}{f(r)} \right\} \tilde{w}_1 \left( \frac{R(x)(y - S(x))}{f(x)} \right)$, respectively. The same bound is established for the first Dirichlet eigenfunction of the Laplacian for $D$. In fact, we treat harmonic functions and eigenfunctions for $L = \Delta_d + \alpha(x, y) \frac{\partial}{\partial x} + \beta(x, y) \nabla_y + h(x, y)$ under suitable conditions on $\alpha$, $\beta$ and $h$. Similar domains were considered in Bañuelos and van den Berg [3]. They treated $L = \Delta_d$ on domains with less regularity than ours and consequently have less sharp bounds.

We obtain sharper bounds in the special case $\Omega = B_{d-1}(0, 1) \subset \mathbb{R}^{d-1}$ and $R(x) \equiv I$, $S(x) \equiv 0$. If $L = \Delta_d + h$ and $Lu = 0$ on $D$, $u > 0$ on $D$, $u(x, 0) \to 0$ as $x \to \infty$ and $u = 0$ on $\partial D \cap \{ (x, y) : x > 0 \}$ or $u$ is the first Dirichlet eigenfunction for $L$ on $D$, we show $u(x, y)$ is bounded within multiplicative constants of $f(x)^{-\frac{d}{2}} \exp \left\{ -\sqrt{\lambda_1} \int_0^x \frac{dr}{f(r)} \right\} w_1 \left( \frac{\|y\|}{f(x)} \right)$ where $w_1$ and $\lambda_1$ are the first Dirichlet eigenfunction and eigenvalue for $\Delta_{d-1}$ on $B_{d-1}(0, 1)$. This
lower bound (when $h \equiv 0$ and $y = 0$) has been recently established by Lindeman, Pang and Zhao [8].

A brief word about the organization of this paper, Section 2 contains statements of our main results which are proved in Section 3. In Section 4 we apply the ideas of Davies and Simon [5] to obtain bounds on heat kernels and other eigenfunctions in terms of bounds on the first eigenfunction when the horn-shaped domain is intrinsically ultracontractive. We also apply the Kelvin transformation to obtain bounds on harmonic functions and eigenfunctions on domains with a cusp. Some of the results we need along the way are found in Section 5, the appendix.

We are indebted to Zhongxin Zhao for pointing out this problem to us.

2 Main Results

In this section we shall state our results and establish notation. Suppose $\Omega \subset \mathbb{R}^{d-1}$ is a bounded domain with $C^{2,\alpha}$ boundary and $f : [0, \infty) \to (0, \infty)$ is a $C^{2,\alpha}$ positive function for which $\lim_{x \to \infty} f(x) = 0$. We will make some additional assumptions on $f$ below (see (2.2)). Let $R : [0, \infty) \to U(d)$ be $C^2$ and $S : [0, \infty) \to \mathbb{R}^{d-1}$ also $C^2$. Define $z(x, y) = \frac{R(x)(y-S(x))}{f(x)}$ for $y = (y_1, \ldots, y^{d-1})$ and finally put $D = \{(x, y) : x > 0, z(x, y) \in \Omega\}$. This is our “horn-shaped” domain.

Given Holder continuous maps $\alpha, h : D \to \mathbb{R}$, and $\beta : D \to \mathbb{R}^{d-1}$ define an operator $L$ by

$$Lu = \Delta_d u + \alpha(x, y) \frac{\partial u}{\partial x} + \beta(x, y) \nabla_y u + h(x, y)u,$$
where $\Delta_d$ is the $d$-dimensional Laplacian. We shall study solutions of

$$
\begin{aligned}
Lu &= 0, \text{ on } D, \\
u &> 0, \text{ on } D, \\
u &= 0, \text{ on } \partial D \cap \{(x, y) : x > 0\}, \\
\lim_{x \to \infty} u(x, y) &= 0.
\end{aligned}
$$

We shall assume

$$
\begin{aligned}
(i) \lim_{x \to \infty} \{ \|f(x)R'(x)\| + \|S'(x)\| \} &= 0, \\
(ii) \lim_{x \to \infty} \{ f^2(x)\|R''(x)\| + f(x)\|S''(x)\| + |f'(x)|\|R'(x)\| \} &= 0, \\
(iii) \lim_{x \to \infty} \left\{ f(x) \left( \sup_{y \in S(x)+f(x)R^{-1}(x)\Omega} |\alpha(x, y)| + \sup_{y \in S(x)+f(x)R^{-1}(x)\Omega} \|\beta(x, y)\| \right) \right\} &= 0, \\
(iv) \lim_{x \to \infty} \{ |f'(x)| + f(x)|f''(x)| \} &= 0, \\
v) \lim_{x \to \infty} \sup_{y \in S(x)+f(x)R^{-1}(x)\Omega} f^2(x)|h(x, y)| &= 0.
\end{aligned}
$$

The conditions (2.2) are satisfied if e.g. $f, f', f''$ all tend to zero at infinity and $\alpha, \beta, h$ are bounded and $R, S$ are $C^2_b$ (bounded derivatives up to 2nd order) functions with $\lim_{x \to \infty} \|S'(x)\| = 0$.

In what follows, $\tilde{w}_1$ and $\tilde{\lambda}_1$ will be the first Dirichlet eigenfunction and eigenvalue of $\Delta_{d-1}$ on $\Omega$, respectively, with $\sup_{\Omega} \tilde{w}_1 = \tilde{w}_1(z_0) = 1$ for some $z_0 \in \Omega$.

Our first result is the following estimate.

**Theorem 2.1.** Suppose $u$ is a solution to $Lu = 0$ in $D$, $u > 0$ on $D$, $u = 0$ on $\partial D \cap \{x > 0\}$, $\lim_{x \to \infty} u(x, y) = 0$ with $L$ satisfying (2.2). Then, for every $\delta \in (0, \tilde{\lambda}_1)$, there exist $x_0 > 0$ and constant $C$ such that

$$C^{-1} e^{-\sqrt{\lambda_1+\delta} \int_0^\infty \frac{dr}{f(r)} \tilde{w}_1(z)} \leq u(x, y) \leq C e^{-\sqrt{\lambda_1-\delta} \int_0^\infty \frac{dr}{f(r)} \tilde{w}_1(z)}, \quad x > x_0, z \in \Omega.$$
Corollary 2.2. Under the conditions of Theorem 2.1,

$$\lim_{x \to \infty} \sup_{y \in S(x)+f(x)\mathbb{R}(x)^{-1}\Omega} \frac{\log(u(x,y))}{\int_0^x \frac{dt}{f(t)}} = -\sqrt{\tilde{\lambda}_1}.$$ 

We now focus on the special case:

$$\Omega = B_{d-1}(0,1) \subset \mathbb{R}^{d-1}$$

$$D = \{(x,y) : x > 0, \|y\| < f(x)\}$$

and

$$Lu = \Delta_d u + hu = 0, \quad \text{in } D$$

$$u > 0, \quad \text{in } D$$

$$u = 0, \quad \text{on } \partial D \cap \{(x,y) : x > 0\}$$

$$\lim_{x \to \infty} \sup_{|y| \leq f(x)} u(x,y) = 0.$$ 

(2.3)

We denote by $w_1$ and $\lambda_1$ the first Dirichlet eigenfunction and eigenvalue for $\Delta_{d-1}$ on $B_{d-1}(0,1)$ with $\sup_{B_{d-1}(0,1)} w_1 = w_1(0) = 1$. We will make the following assumption on $f$.

$$\left\{ \begin{array}{l}
(i) \quad \text{The condition (v) of assumption (2.2) is satisfied}, \\
(ii) \quad f \sup_{|y| \leq f(x)} |h(x,y)| \in L^1[0,\infty), \\
(iii) \quad \frac{f}{f'2} \in L^1[0,\infty), \\
(iv) \quad f'' \in L^1[0,\infty), \\
(v) \quad \lim_{x \to \infty} f^2 \cdot f'''(x) = 0 \quad \text{and} \quad ff'' \in L^1[0,\infty), \\
(vi) \quad \lim_{x \to \infty} f^3 \cdot f''(x) = 0 \quad \text{and} \quad f^2 f''' \in L^1[0,\infty). 
\end{array} \right.$$ 

(2.4)

Note that the condition (iv) of assumption (2.2) can be easily derived from (2.4) (iv) and (v). Then we shall establish
Theorem 2.3. If \( u \) solves (2.3) and \( f \) and \( h \) satisfy (2.4), then there is some positive constant \( C \) such that

\[
C^{-1}f(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)} \right\} w_1 \left( \frac{\|y\|}{f(x)} \right) \leq u(x, y) \leq Cf(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)} \right\} w_1 \left( \frac{\|y\|}{f(x)} \right),
\]

for \((x, y) \in D, x > 1\).

3 Proofs of Main Results

We will use the following test function in the proof of Theorem 2.1,

\[
u(x, y) = \exp \left\{ -c \int_0^x \frac{d\tau}{f(\tau)} \right\} w(z),
\]

where \( z = z(x, y) \). Notice that if we denote the derivative of \( z \) with respect to \( \cdot \) by \( Dz \), then

\[
\begin{aligned}
D_x z &= \frac{R'(y - S)}{f} - \frac{RS'}{f} - \frac{f'}{f} z, \\
D_x^2 z &= \frac{R''(y - S)}{f} - \frac{2R'S'}{f^2} - \frac{2f'R'(y - S)}{f^2} + \frac{2f''RS'}{f^2} - \frac{RS''}{f} + \frac{2(f')^2}{f^2} z - \frac{f''}{f} z, \\
\frac{\partial z^i}{\partial y^j} &= \frac{R^i_j}{f}, \\
\frac{\partial^2 z^i}{\partial y^j \partial y^k} &= 0,
\end{aligned}
\]

(3.1)

Then, by (3.1) we have

\[
Lu = f^{-2} \exp \left\{ -c \int_0^x \frac{d\tau}{f(\tau)} \right\} \left\{ \Delta_{d-1} w + c^2 w + D_z^2 w(fD_z z, fD_x z) + \langle \nabla_z w, f^2D_z^2 z \rangle \\
+ \alpha f \langle \nabla_z w, fD_x z \rangle + \langle f\beta, \nabla_z w fD_y z \rangle + (cf' + f^2 h + cf\alpha)w \right\},
\]

(3.2)
Define

\[ \tilde{A}w = D_z^2 w(fD_x z, fD_y z) + \langle \nabla_z w, f^2 D_z^2 z \rangle + \alpha f \langle \nabla w, fD_x z \rangle + \langle f \beta, \nabla w fD_y z \rangle + (cf' + f^2 h + cf \alpha) w \]

so that

\[ Lu = f^{-2} \exp \left\{ -c \int_0^x \frac{d\tau}{f(\tau)} \right\} \{ \Delta_{d-1} w + c^2 w + \tilde{A}w \} . \]

Under (2.2) the operator

\[ \mathcal{A} = \Delta_{d-1} + c^2 + \tilde{A} \]

is a small perturbation of \( \Delta_{d-1} + c^2 \).

Denote by \( \mathbf{n} \) the unit outward pointing normal on \( \partial \Omega \) and for \( \xi \in \partial \Omega \) put \( x(t, \xi) = \xi - t \mathbf{n} \), \( t \geq 0 \).

Since \( \partial \Omega \) is compact and \( C^2 \), there is an \( \eta' > 0 \) such that \( x(t, \xi), 0 \leq t \leq \eta' \) satisfies \( x(t, \xi) = x(s, \zeta) \) if and only if \( t = s \) and \( \xi = \zeta \). Also, given \( \delta > 0 \) small enough, there is an \( \eta'' \) such that

\[ -(\tilde{w}_1(x(t, \xi)))' > \delta \quad \text{for } 0 \leq t < \eta'' . \]

This follows readily from the Hopf maximum principle (Lemma 3.4 of [7]). Put \( \eta = \eta' \wedge \eta'' \), and take \( \varphi \) to be a smooth function, \( \varphi : [0, \infty) \to [0, 1] \) with \( \varphi(t) = 1 \) for \( 0 \leq t \leq \eta/2 \), \( \varphi(t) = 0 \) for \( t > \eta \) and \( \varphi \) monotone decreasing on \( [\eta/2, \eta] \). Define \( \psi : \Omega \to [0, 1] \) by setting \( \psi(y) = \varphi(t) \) if \( y = x(t, \xi) \) for some \( \xi \in \partial \Omega \), \( 0 < t < \eta \) and putting \( \psi(y) = 0 \) otherwise.

Set, for \( a > 0 \), \( \Omega(a) = \{ x(t, \xi) : \xi \in \partial \Omega, 0 < t < a \} \) and extend \( \mathbf{n} \) to denote the unit tangent field to the flow \( x(\cdot, \xi) \) on \( \Omega(\eta') \). Now given \( \epsilon > 0 \), let \( \tilde{w}_1^{\pm \epsilon} \) and \( \tilde{\lambda}_1^{\pm \epsilon} \) be the first Dirichlet eigenfunction and eigenvalue, respectively, for \( \mathcal{A}^{\pm \epsilon} = \Delta_{d-1} \mp \epsilon \psi \mathbf{n} \cdot \nabla \) on \( \Omega \) with \( \sup_\Omega \tilde{w}_1^{\pm \epsilon} = 1 \). By Schauder estimates (Theorem 6.6 of [7]), there is a \( C_1 \) independent of
\( \epsilon \) such that for \( \epsilon \in (0, 1) \), and \( |u|_{2,0,\Omega} \equiv \sum_{j=0}^{2} \sup_{\Omega} \sup_{|\beta|=j} |D^\beta u| \), \( D^\beta = \partial_x^\beta_1 \partial_y^\beta_2 \ldots \partial_y^{\beta_{d-1}}, \)

\[ |\beta| = \sum_1^d \beta_i, \]

\[ |\tilde{w}_1|_{2,0,\Omega} + |\tilde{w}_1^{\pm \epsilon}|_{2,0,\Omega} \leq C_1 \sup \{ \sup_{\Omega} \tilde{w}_1, \sup_{\Omega} \tilde{w}_1^{\pm \epsilon} \} = C_1. \]

Then the uniqueness of the first eigenfunction implies that \( \tilde{\lambda}_1^{\pm \epsilon} \to \tilde{\lambda}_1 \) and \( \sup_{\Omega} |\tilde{w}_1^{\pm \epsilon} - \tilde{w}_1| \to 0 \) as \( \epsilon \to 0 \).

Applying \( A^{\pm \epsilon} \) to \( \tilde{w}_1^{\pm \epsilon} - \tilde{w}_1 \) yields

\[ A^{\pm \epsilon}(\tilde{w}_1^{\pm \epsilon} - \tilde{w}_1) = (\tilde{\lambda}_1 - \tilde{\lambda}_1^{\pm \epsilon})\tilde{w}_1^{\pm \epsilon} + \tilde{\lambda}_1(\tilde{w}_1^{\pm \epsilon} - \tilde{w}_1) \pm \epsilon \psi \tilde{\eta} \nabla \tilde{w}_1 \]

and so by Schauder estimates (Theorem 6.6 [7]),

\[ |\tilde{w}_1^{\pm \epsilon} - \tilde{w}_1|_{2,0,\Omega} \leq C_2( |\tilde{\lambda}_1 - \tilde{\lambda}_1^{\pm \epsilon}| + \sup_{\Omega} |\tilde{w}_1^{\pm \epsilon} - \tilde{w}_1| + \epsilon ) \]

which implies \( |\tilde{w}_1^{\pm \epsilon} - \tilde{w}_1|_{2,0,\Omega} \to 0 \).

From (2.2) and the above estimates, we have

\[ \sup_{\Omega} |\tilde{A}\tilde{w}_1^{\pm \epsilon}| = o(1) \text{ as } x \to \infty . \]

In addition, from (3.4), we deduce there is an \( \epsilon_0(\delta/2) \) such that

\[ -\psi \tilde{\eta} \cdot \nabla \tilde{w}_1^{\pm \epsilon} \geq \delta/2 \text{ on } \Omega(\eta/2), \epsilon < \epsilon_0(\delta/2) . \]

\[ -\psi \tilde{\eta} \nabla \tilde{w}_1^{\pm \epsilon} \geq 0 \text{ on } \Omega, \epsilon < \epsilon_0(\delta/2) . \]

Let \( C_3 = \inf_{0<\epsilon<\epsilon_0(\delta/2)} \inf_{\Omega(\eta/2)} |\tilde{w}_1^{\pm \epsilon}| \) and notice that \( 1 > C_3 > 0 \) by Harnack’s inequality.

From the fact that \( \tilde{\lambda}_1^{\pm \epsilon} \to \tilde{\lambda}_1 \), we see there is an \( \epsilon_1(\delta, C_4) \) such that

\[ |\tilde{\lambda}_1^{\pm \epsilon} - \tilde{\lambda}_1| < \delta/2 , 0 < \epsilon < \epsilon_1(\delta, C_4) . \]

Finally fix \( \epsilon = \epsilon_0(\delta) \wedge \epsilon_1(\delta, C_4) \wedge 1 \). Then there is an \( x_0(\delta, \epsilon) \) such that

\[ \sup_{\Omega} |\tilde{A}\tilde{w}_1^{\pm \epsilon}| \leq C_4 \epsilon \delta/2 , x > x_0(\delta, \epsilon) . \]

We can now prove
Proposition 3.1. There exist positive $\epsilon, \delta, x_0$ such that

$$u_-(x, y) = \exp \left\{ -\sqrt{\tilde{\lambda}_1 + \delta} \int_0^x \frac{d\tau}{f(\tau)} \right\} \tilde{w}_1^{-\epsilon}(z)$$

and

$$u_+(x, y) = \exp \left\{ -\sqrt{\tilde{\lambda}_1 - \delta} \int_0^x \frac{d\tau}{f(\tau)} \right\} \tilde{w}_1^{+\epsilon}(z)$$

are sub- and super-solutions, respectively, for $L$ on $D \cap \{(x, y) : x > x_0\}$.

Proof. Take $\epsilon, \delta, x_0$ as in the discussion preceding the Proposition. We give the proof for $u_-$, that for $u_+$ being entirely analogous. Now

$$Lu_- = f^{-2} \exp \left\{ -\sqrt{\tilde{\lambda}_1 + \delta} \int_0^x \frac{d\tau}{f(\tau)} \right\} A\tilde{w}_1^{-\epsilon}(z),$$

with

$$A\tilde{w}_1^{-\epsilon}(z) = (\tilde{\lambda}_1 + \delta - \tilde{\lambda}_1^{-\epsilon})\tilde{w}_1^{-\epsilon}(z) - \epsilon \psi \tilde{n} \cdot \nabla \tilde{w}_1^{-\epsilon}(z) + \tilde{A}\tilde{w}_1^{-\epsilon}(z).$$

Now, on $\Omega(\eta/2)$, we have by (3.5), (3.7) and (3.8) that

$$-\epsilon \psi \tilde{n} \cdot \nabla \tilde{w}_1^{-\epsilon} + \tilde{A}\tilde{w}_1^{-\epsilon} \geq 0, \ x > x_0$$

and

$$(\tilde{\lambda}_1 + \delta - \tilde{\lambda}_1^{-\epsilon})\tilde{w}_1^{-\epsilon} \geq 0.$$ 

On $\Omega \setminus \Omega(\eta/2)$ we have by (3.6), (3.7) and (3.8) that

$$(\tilde{\lambda}_1 + \delta - \tilde{\lambda}_1^{-\epsilon})\tilde{w}_1^{-\epsilon} + \tilde{A}\tilde{w}_1^{-\epsilon} \geq 0, \ x > x_0$$

and

$$-\epsilon \psi \tilde{n} \cdot \nabla \tilde{w}_1^{-\epsilon} \geq 0.$$ 

Thus $A\tilde{w}_1^{-\epsilon} \geq 0$ and $u_-$ is a sub-solution as claimed. The proof that $u_+$ is a super-solution is entirely analogous
Finally, we note that

**Lemma 3.2.** There exist constants $\epsilon_0, C$, so that for all $\epsilon < \epsilon_0$

$$ C^{-1} \tilde{w}_1^\pm(z) \leq \tilde{w}_1(z) \leq C \tilde{w}_1^\pm(z), \quad z \in \Omega. $$

**Proof.** The Hopf maximum principle (Lemma 3.4 [7]) gives a $\gamma > 0$ so that the inequality holds in $\Omega(\gamma)$. For $z \in \Omega \setminus \Omega(\gamma)$ it follows from Harnack’s inequality.

**Proof of Theorem 2.1.** Combine Proposition 3.1, Lemma 3.2 and the maximum principle.

Now consider the special case $\Omega = B(0, 1) \subset \mathbb{R}^{d-1}, \alpha = 0, \beta = 0, R(x) \equiv I$ and $S(x) \equiv 0$. Then $w_1(z) = w_1(y/f(x))$ is a function of $t = \|y\|/f(x)$ and (3.2) becomes

$$ Lu = f^{-2} \exp \left\{ -c \int_0^x \frac{d\tau}{f(\tau)} \right\} \left\{ w''(t) + \frac{1}{1 + t^2 f''} \left( \frac{d - 2}{t} + t(2cf' + 2f'' - f'') \right) w'(t) + \frac{1}{1 + t^2 f''}(c^2 + cf' + f^2 h)w(t) \right\}. $$

We will now improve the bound of Theorem 2.1 by proving Theorem 2.3 under additional assumptions on $f$. However, Theorem 2.1 will be valid in this setting if the following condition is satisfied

$$ \lim_{x \to \infty} \{ |f'(x)| + f'^2(x) + f(x)|f''(x)| + f^2(x) \sup_{|y|<f(x)} |h(x, y)| \} = 0. $$

We also observe that a moving plane argument shows that (please see Lemma 5.1 in the appendix for details) if $h(x, y) = h(x, \|y\|)$ with $h$ nonincreasing in $\|y\|$ then

$$ u(x, 0) = \max_{|y|<f(x)} u(x, y) \text{ if } u(0, y) = u(0, \|y\|) \text{ is also nonincreasing in } \|y\|. $$

Note both $\{f, h^+(x) = \sup_{|y|\leq f(x)} h(x, y)\}$ and $\{f, h^-(x) = \inf_{|y|\leq f(x)} h(x, y)\}$ satisfy (2.4) when $\{f, h\}$ does. Let $x_1 > 0$ be given by Lemma 5.3 (see Appendix). By the Hopf
boundary lemma (Lemma 3.4 of [7]) there exist \(0 < c_1 < c_2\) such that

\[
c_1 w_1\left(\frac{\|y\|}{f(x_1)}\right) \leq u(x_1, y) \leq c_2 w_1\left(\frac{\|y\|}{f(x_1)}\right).
\]

And it is clear that both \(\{f, h^+\}\) with \(c_2 w_1\left(\frac{\|y\|}{f(x_1)}\right)\) and \(\{f, h^-\}\) with \(c_1 w_1\left(\frac{\|y\|}{f(x_1)}\right)\) satisfy Lemma 5.3. Let \(u^+\) and \(u^-\) be the solutions of (5.3) for boundary data \(c_2 w_1\left(\frac{\|y\|}{f(x_1)}\right)\) and \(c_1 w_1\left(\frac{\|y\|}{f(x_1)}\right)\), respectively, on \(\tilde{D}_{x_1}\). Then Lemma 5.2 implies that

\[
u \leq u \leq u^+ \quad \text{on } D \cap \{(x, y): x > x_1\}.
\]

Furthermore it is clear that Lemma 5.1 applies to both \(u^+\) and \(u^-\). Hence it is sufficient to prove Theorem 2.3 for \(u^+\) and \(u^-\) with \(\{f, h^+\}\) and \(\{f, h^-\}\), respectively. Therefore from now on in this section we shall assume that \(u\) solves (2.3) with

\[
u(x, y) = u(x, r), \; h(x, y) = h(x), \; r = \|y\| \quad \text{and } u \text{ is decreasing in } r.
\]

The proof relies on the following change of variables,

\[
t = \frac{r}{f(x)} \quad (t \in [0, 1]), \quad s = \int_0^x \frac{d\tau}{f(\tau)} + \frac{1}{2} t^2 f'(x).
\]

Given \(s\) and \(t\), this defines \(x = x(s, t)\) and for future use we record the formulas

\[
\frac{\partial x}{\partial s}(s, t) = \frac{f}{1 + \frac{1}{2} t^2 f''},
\]

\[
\frac{\partial x}{\partial t}(s, t) = \frac{-tf'f''}{1 + \frac{1}{2} t^2 f''},
\]

\[
\frac{\partial^2 x}{\partial t^2}(s, t) = \frac{-ff' + t(ff'' + f'')}x_t}{1 + \frac{1}{2} t^2 f''} + \frac{ff''(ff'' + \frac{1}{2} t^2 (f'f'' + f f'')x_t)}{(1 + \frac{1}{2} t^2 f'')^2},
\]

\[
\frac{\partial^2 x}{\partial s \partial t}(s, t) = \frac{f'x_t}{1 + \frac{1}{2} t^2 f''} - \frac{ff''(ff'' + \frac{1}{2} t^2 (f'f'' + f f'')x_t)}{(1 + \frac{1}{2} t^2 f'')^2}.
\]

From now on, \(f, f', f'',\) etc., and \(h\) will be evaluated at \(x(s, t)\). Then if \(u\) is a solution of (2.3), write

\[
u(x, y) = u(x, r) = f(x)^{\frac{2-d}{2}} v(s, t).
\]
Then, (2.3) forces $v$ to satisfy

$$Av(s, t) \equiv (1 - t^2 f'(x)^2 + t^4 f'(x)^4 + t^2 f(x) f''(x) - t^4 f(x) f'(x)^2 f''(x)$$

$$+ \frac{1}{4} t^4 f(x)^2 f''(x)^2 \frac{\partial^2 v}{\partial s^2} (s, t) + (1 + t^2 f'(x)^2) \frac{\partial^2 v}{\partial t^2} (s, t)$$

$$+ \left( \frac{d - 2}{t} + dt f'(x)^2 - tf(x) f''(x) \right) \frac{\partial v}{\partial t} (s, t)$$

$$+ (2t^3 f'(x)^3 - t^3 f(x) f'(x) f''(x)) \frac{\partial^2 v}{\partial s \partial t} (s, t)$$

$$+ (t^2 f'(x)^3 + dt f'(x)^3 - 2t^2 f(x) f'(x) f''(x)$$

$$- \frac{1}{2} dt^2 f(x) f'(x) f''(x) + \frac{1}{2} t^2 f(x)^2 f^{(3)}(x)) \frac{\partial v}{\partial s} (s, t)$$

$$+ (f(x)^2 h(x) + \frac{d}{2} \left( \frac{d}{2} - 1 \right) f'(x)^2 + \left( 1 - \frac{d}{2} \right) f(x) f''(x)) v(s, t)$$

$$= 0.$$  

After dividing everything by the coefficient of $\frac{\partial^2 v}{\partial s^2}$, this equation is written in the following suggestive manner.

$$Av(s, t) \equiv \frac{\partial^2 v}{\partial s^2} (s, t) + \frac{\partial^2 v}{\partial t^2} (s, t) + \frac{d - 2}{t} \frac{\partial v}{\partial t} (s, t)$$

$$+ \tilde{c}_1(s, t) \frac{\partial^2 v}{\partial t^2} (s, t) + \tilde{c}_2(s, t) \frac{\partial^2 v}{\partial s \partial t} (s, t) + \tilde{c}_3(s, t) \frac{\partial v}{\partial t} (s, t)$$

$$+ \tilde{c}_4(s, t) \frac{\partial v}{\partial s} (s, t) + \tilde{c}_5(s, t) v(s, t) = 0$$

with

$$k(s, t) = ff'' - f'^2,$$

$$\ell(s, t) = t^2 k(1 - t^2 f'^2) + \frac{1}{4} t^4 f^2 f'^2,$$

$$\tilde{c}_1(s, t) = \frac{t^2 f'^2 - \ell}{1 + \ell},$$

$$\tilde{c}_2(s, t) = \frac{t^3 (2f'^3 - ff'f'')}{1 + \ell},$$

$$\tilde{c}_3(s, t) = -t(df'^2 - ff'') - (d - 2) t^{-1} \ell,$$

$$\tilde{c}_4(s, t) = \frac{(d + 1)t^2 f'^3 - \frac{1}{2}(d + 4) t^2 ff'f'' + \frac{1}{2} t^2 f^2 f'''}{1 + \ell},$$

$$\tilde{c}_5(s, t) = 0.$$
(3.23) \[ \tilde{c}_5(s, t) = \frac{f^2 h + \frac{1}{4} d(d-2) f'^2 - \frac{1}{2}(d-2) ff''}{1 + \ell} . \]

Assumption (2.4) insures that for \( s_0 \) large enough \( |\ell(s, t)| \leq \frac{1}{2} \) for \( s \geq s_0, 0 \leq t < 1. \) In fact, \( \lim_{s \to \infty} \sup_{|t| < 1} |\ell(s, t)| = 0. \) Thus \( \lim_{s \to \infty} \sup_{0 \leq t \leq 1} |\tilde{c}_i(s, t)| = 0 \) for \( i = 1, 2, 3, 4, 5. \)

If the coefficients \( \tilde{c}_i(s, t) \) were identically zero then we would be considering \( v = \tilde{v} \) with \( \tilde{A}\tilde{v} = \frac{\partial^2 v}{\partial s^2} + \frac{\partial^2 \tilde{v}}{\partial t^2} + \frac{d-2}{t} \frac{\partial \tilde{v}}{\partial t} = 0. \) A solution to the latter equation suited to the boundary conditions in (2.3) is \( \tilde{v}(s, t) = e^{-\sqrt{\lambda_1} s} w_1(t). \) This motivates the bounds in Theorem 2.3. Our strategy to obtain the estimate in Theorem 2.3 when the \( \tilde{c}_i \neq 0 \) is to define

\[ \varphi_1(s) = \int_0^1 w_1(t) t^{d-2} v(s, t) dt \]

and first compare \( v(s, t) \) to \( \varphi_1(s) w_1(t) \). Then we obtain estimates on \( \varphi_1 \) from examining the differential equation it must satisfy because of (3.16).

We have the following

**Proposition 3.3.** There is an \( N_1 > 1 \) such that

\[ N_1^{-1} \sup_{0 \leq t \leq 1} v(s, t) \leq \varphi_1(s) \leq N_1 \sup_{0 \leq t \leq 1} v(s, t) . \]

**Proof.** The right-hand inequality is automatic. The left-hand inequality follows from Harnack’s inequality and the fact that \( u(x, r) \) is decreasing in \( r. \) \( \square \)

**Proposition 3.4.** There is an \( N_2 \geq N_1, s_0 > 0, \) such that for \( s > s_0, \)

\[ N_2^{-1} \varphi_1(s) w_1(t) \leq v(s, t) \leq N_2 \varphi_1(s) w_1(t) . \]

**Proof.** First select \( s_0 = \max\{1, \int_0^{x_1^1} \frac{f'(r)}{f(r)} + \frac{1}{2} |f'(x_1)|\} \) where \( x_1 \) is given in Lemma 5.3, we shall make \( s_0 \) larger for the left-hand inequality.

By Schauder estimates (recall the norms from Section 2),

\[ |v|_{2, 0; [s-1/2, s+1/2] \times [0, 1]} \leq C |v|_{0, 0; ([s-1, s+1] \times [0, 1])} \]

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the right-hand side is bounded by $CN_1\varphi_1(s)$ by Harnack’s inequality, (5.4), the fact that $u(x,r)$ is decreasing in $r$ and Proposition 3.3. Since $v(s,1) = 0$, $v(s,t) \leq C_3(1-t)\varphi_1(s)$, for some $C_3 > 0$. But $w_1(t) \geq c_4^{-1}(1-t)$ which implies the right-hand side.

For the lower bound, we use a well-known technique due to Hopf. By Proposition 3.3, and Harnack’s inequality, there is a positive constant $c_5$ so that

$$c_5\varphi(s) \leq v(s,t) , \quad 0 \leq t \leq 3/4 .$$

(3.24) Define $\xi(t) = e^{-\alpha t^2} - e^{-\alpha}$. Then

$$\mathcal{A}\xi(t) = e^{-\alpha t^2}[2\alpha(2\alpha - 1)(1 + \tilde{c}_1(s,t)) - 2\alpha((d - 2) + t\tilde{c}_3(s,t))]
\quad + \tilde{c}_5(s,t)\xi(t)
\quad \leq e^{-\alpha t^2}[2\alpha(2\alpha - 1)(1 + \tilde{c}_1(s,t)) - 2\alpha((d - 2) + t\tilde{c}_3(s,t)) - \tilde{c}_5^- (s,t)] .$$

Since $\tilde{c}_1 = o(1)$, $\tilde{c}_3 = o(1)$ and $\tilde{c}_5^- = o(1)$ as $s \to \infty$, if $s_0$ is large and $s \geq s_0$, we can select $\alpha$ large enough so that

$$\mathcal{A}\xi(t) \geq 0 , \quad 3/4 \leq t \leq 1 .$$

(3.25) By (2.24), (2.25) and the maximum principle; there is a constant $c_6$ such that

$$c_6\varphi(s)\xi(t) \leq v(s,t) , \quad 3/4 \leq t < 1 .$$

But there is a $c_7 > 0$ so that

$$\xi(t) \geq c_7(1-t) , \quad 3/4 \leq t < 1 .$$

Since $(1-t) \geq c_8 w(t)$, for some $c_8 > 0$ and $3/4 \leq t < 1$, the proof is complete. \hfill \Box

We now multiply both sides of (3.16) by $w_1(t)t^{d-2}$ and integrate from $t = 0$ to $t = 1$. 

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The resulting equation, \( \int_0^1 \mathcal{A}v(s,t)w_1(t)t^{d-2}dt = 0 \), after integrations by parts, becomes

\[
\varphi''_1(s) - \lambda_1 \varphi_1(s) + \int_0^1 \frac{d}{dt^2}(c_1(s,t)w_1(t)t^{d-2})v(s,t)dt
- \int_0^1 \frac{d}{dt}(c_2(s,t)w_1(t)t^{d-2}) \frac{\partial v}{\partial s}(s,t)dt - \int_0^1 \frac{d}{dt}(c_3(s,t)w_1(t)t^{d-2})v(s,t)dt
+ \int_0^1 \tilde{c}_4(s,t)w_1(t)t^{d-2} \frac{\partial v}{\partial s}(s,t)dt + \int_0^1 \tilde{c}_5(s,t)v(s,t)w_1(t)t^{d-2}dt = 0 .
\]

Introducing

\[
\begin{align*}
g_1(s,t) &= \frac{d^2}{dt^2}(c_1(s,t)w_1(t)t^{d-2}) \\
g_2(s,t) &= -\frac{d}{dt}(c_2(s,t)w_1(t)t^{d-2}) \\
g_3(s,t) &= -\frac{d}{dt}(c_3(s,t)w_1(t)t^{d-2}) \\
g_4(s,t) &= \tilde{c}_4(s,t)w_1(t)t^{d-2} \\
g_5(s,t) &= \tilde{c}_5(s,t)w_1(t)t^{d-2} .
\end{align*}
\]

Then (3.26) can be written

\[
\varphi''_1(s) - \lambda_1 \varphi_1(s) + \int_0^1 (g_1(s,t) + g_3(s,t) + g_5(s,t))v(s,t)dt
+ \int_0^1 (g_2(s,t) + g_4(s,t)) \frac{\partial v}{\partial s}(s,t)dt = 0 .
\]

**Lemma 3.5.** Under assumption (2.4)

\[\varphi'_1(s) < 0 \text{ at } \infty .\]

**Proof.** From Proposition 3.4 and (5.4) we have for some \( C \) that

\[
C^{-1}e^{-\frac{3\sqrt{s_1}}{2}(s-s_1)}\varphi_1(s)w_1(t) \leq v(s,t) \leq Ce^{-\frac{\sqrt{s_1}}{2}(s-s_1)}\varphi_1(s)w_1(t) \quad s \geq s_1 \geq s_0 ,
\]

and then

\[
C^{-1}e^{-\frac{3\sqrt{s_1}}{2}(s-s_1)}\varphi_1(s) \leq \varphi_1(s) \leq Ce^{-\frac{\sqrt{s_1}}{2}(s-s_1)}\varphi_1(s) \quad s \geq s_1 \geq s_0 .
\]

Next integrate (3.28) from \( s \) to \( \infty \), obtaining

\[
\begin{align*}
\varphi'_1(s) &= -\lambda_1 \int_s^\infty \varphi_1(\sigma)d\sigma + \int_s^\infty \int_0^1 (g_1(\sigma,t) + g_3(\sigma,t) + g_5(\sigma,t))v(\sigma,t)dtd\sigma \\
&\quad + \int_s^\infty \int_0^1 (g_2(\sigma,t) + g_4(\sigma,t)) \frac{\partial v}{\partial \sigma}(\sigma,t)dtd\sigma.
\end{align*}
\]

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Integration by parts gives

$$
\varphi_1'(s) = -\lambda_1 \int_s^\infty \varphi_1(\sigma) d\sigma + \int_s^\infty \int_0^1 (g_1(\sigma, t) + g_3(\sigma, t) + g_5(\sigma, t)) v(\sigma, t) dt d\sigma
$$

(3.32)

$$
- \int_0^1 (g_2(s, t) + g_4(s, t)) v(s, t) dt - \int_s^\infty \int_0^1 \frac{\partial(g_2 + g_4)}{\partial \sigma}(\sigma, t) v(\sigma, t) dt d\sigma .
$$

By (3.4) and (3.30) we obtain that for \( s \geq s_0 \)

$$
\begin{align*}
\varphi_1'(s) &\leq -\lambda_1 \int_s^\infty C^{-1} e^{-\frac{3\sqrt{\lambda}}{2}(\sigma-s)} \varphi_1(\sigma) d\sigma \\
&+ \int_s^\infty \int_0^1 |g_1(\sigma, t) + g_3(\sigma, t) + g_5(\sigma, t)| C e^{-\frac{3\sqrt{\lambda}}{2}(\sigma-s)} \varphi_1(s) w_1(t) dt d\sigma \\
&+ \int_0^1 |g_2(s, t) + g_4(s, t)| C e^{-\frac{3\sqrt{\lambda}}{2}(\sigma-s)} \varphi_1(s) w_1(t) dt \\
&+ \int_s^\infty \int_0^1 |\frac{\partial(g_2 + g_4)}{\partial \sigma}(\sigma, t)| C e^{-\frac{3\sqrt{\lambda}}{2}(\sigma-s)} \varphi_1(s) w_1(t) dt d\sigma,
\end{align*}
$$

(3.33)

that is

$$
\begin{align*}
\varphi_1(s) &\leq C \varphi_1(s) \left( -\frac{2\sqrt{\lambda}}{3c^2} + \int_0^1 |g_2(s, t) + g_4(s, t)| w_1(t) dt \\
&+ \int_s^\infty \int_0^1 (|g_1(\sigma, t) + g_3(\sigma, t) + g_5(\sigma, t)| + |\frac{\partial(g_2 + g_4)}{\partial \sigma}(\sigma, t)|) \\
&\quad e^{-\frac{3\sqrt{\lambda}}{2}(\sigma-s)} w_1(t) dt d\sigma \right) .
\end{align*}
$$

(3.34)

But arguments similar to the ones in the proof of Lemma 3.6 below show that

$$
\lim_{s \to \infty} \left\{ \int_0^1 |g_2(s, t) + g_4(s, t)| w_1(t) dt \\
&+ \int_s^\infty \int_0^1 (|g_1(\sigma, t) + g_3(\sigma, t) + g_5(\sigma, t)| + |\frac{\partial(g_2 + g_4)}{\partial \sigma}(\sigma, t)|) \\
&\quad e^{-\frac{3\sqrt{\lambda}}{2}(\sigma-s)} w_1(t) dt d\sigma \right\} = 0 .
$$

(3.35)

This completes the proof. \( \square \)

Next multiply (3.28) by \( \varphi_1'(s) \) and integrate from \( s \) to \( \infty \), obtaining

$$
\begin{align*}
&-\frac{1}{2} \varphi_1'(s)^2 + \frac{1}{2} \lambda_1 \varphi_1(s)^2 \\
&+ \int_s^\infty \int_0^1 (g_1(\sigma, t) + g_3(\sigma, t) + g_5(\sigma, t)) v(\sigma, t) \varphi_1'(\sigma) d\sigma dt \\
&+ \int_s^\infty \int_0^1 (g_2(\sigma, t) + g_4(\sigma, t)) \frac{\partial w}{\partial \sigma}(\sigma, t) \varphi_1'(\sigma) dt d\sigma = 0 .
\end{align*}
$$

(3.36)
Define

\begin{align}
K(s) &= \int_s^\infty \int_0^1 \left( g_1(\sigma, t) + g_3(\sigma, t) + g_5(\sigma, t) \right) \psi(\sigma, t) \varphi'_1(\sigma) d\sigma dt \\
L(s) &= \int_s^\infty \int_0^1 \left( g_2(\sigma, t) + g_4(\sigma, t) \right) \frac{\partial \psi}{\partial \sigma}(\sigma, t) \varphi'_1(\sigma) d\sigma dt .
\end{align}

(3.37)

(3.38)

Thus, from (3.36), (3.37) and (3.38) we get

\begin{equation}
\frac{\varphi'_1(s)}{\varphi_1(s)} = -\sqrt{\lambda_1 + 2 \frac{K(s)}{\varphi^2_1(s)} + 2 \frac{L(s)}{\varphi^2_1(s)}}
\end{equation}

(3.39)

from which it is clear that we need estimates on \( K(s) \) and \( L(s) \). In particular, we shall show first that

**Lemma 3.6.** Under assumption (2.4)

\( \lim_{s \to \infty} \frac{K(s)}{\varphi^2_1(s)} = \lim_{s \to \infty} \frac{L(s)}{\varphi^2_1(s)} = 0. \)

**Proof.** We first check that under assumption (2.4)

\begin{equation}
\lim_{s \to \infty} \sup_{0 \leq t \leq 1} |g_i(s, t)| = 0 \quad i = 1, 3, 5.
\end{equation}

(3.40)

For \( i = 1, \)

\( g_1(s, t) = \frac{d^2}{dt^2} (c_1(s, t)w_1(t) t^{d-2}) \)

and since \( w_1(t) t^{d-2} \) is bounded along with its first and second derivatives, it suffices to check that

\begin{equation}
\lim_{s \to \infty} \sup_{0 \leq t \leq 1} \left| \frac{d^k}{dt^k} c_1(s, t) \right| = 0, \quad k = 0, 1, 2.
\end{equation}

(3.41)

For \( k = 0 \) this follows easily. For \( k = 1 \)

\( \frac{d}{dt} c_1(s, t) = \frac{2tf'^2 + 2t^2f'f''x_t - \ell t}{1 + \ell} - \frac{(t^2f'^2 - \ell)\ell t}{(1 + \ell)^2} \)
with
\[
\ell_t = 2tk + t^2k_t - 4t^3kf'^2 - t^4k_t f'^2 - 2t^4kf'f''x_t \\
+ t^3f'^2 + \frac{1}{2}t^4ff'f'^2x_t + \frac{1}{2}t^4f^2f''f'''x_t
\]
\[
k_t = (ff'' - f'f'')x_t.
\]

One sees using (2.4) and (3.12) that \( \lim_{s \to \infty} \sup_{0 \leq t \leq 1} k_t(s, t) = 0 \), so by inspection \( \lim_{s \to \infty} \sup_{0 \leq t \leq 1} \ell_t = 0 \) as well. Thus (2.4) implies (3.41) with \( k = 1 \).

For (3.41) with \( k = 2 \), we write down the rather unattractive expression
\[
\frac{d^2\tilde{c}_1}{dt^2}(s, t) = \frac{2f'^2 + 4tf'fx_t + 4tf'f''x_t + 2tf'^2x_t^2 + 2tf'f'(3)x_t^2 - \ell_{tt}}{(1 + \ell)} \\
- \frac{(2tf'^2 + 2t^2f'f''x_t - \ell_t)\ell_t}{(1 + \ell)} \\
- \frac{(2tf'^2 + 2t^2f'f''x_t - \ell_t)\ell_t + (t^2f'^2 - \ell)\ell_{tt}}{(1 + \ell)^2} \\
+ \frac{2(t^2f'^2 - \ell)\ell_t^2}{(1 + \ell)^3}
\]
with
\[
\ell_{tt} = 2k + 4tk_t + t^2k_{tt} - (4t^3kf'^2)_t - 4t^3k_t f'^2 - t^4k_t f'^2 - 2t^4k_t f'f''x_t \\
- (2t^4kf'f''x_t)_t + (t^3f'^2f''^2)_t + \frac{1}{2}(t^4ff'f''^2x_t)_t \\
+ \frac{1}{2}(f'^2f'^3x_t)_t
\]
and
\[
k_{tt} = (ff^{(4)} - f'^2)x_t^2 + (ff^{(3)} - f'f'')x_{tt}.
\]

Now a simple check using the above, (3.12), (3.13), (3.17) and (3.18), reveals that (2.4) implies (3.41) for \( k = 2 \).

For \( i = 3 \),
\[
g_3(s, t) = -(w_1(t)t^{d-2})'\tilde{c}_3(s, t) - w_1(t)t^{d-2}d\tilde{c}_3 \]
\[
(s, t),
\]
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so we only need check

\[(3.42) \quad \lim_{s \to \infty} \sup_{0 \leq t \leq 1} \bar{c}_3(s, t) = 0 ,\]

and

\[(3.43) \quad \lim_{s \to \infty} \sup_{0 \leq t \leq 1} \left| \frac{d\bar{c}_3}{dt}(s, t) \right| = 0 .\]

That (3.42) holds under (2.4) is easily verified. As for (3.43), the only possible difficulty arises in \( \frac{(d-2)t}{t} \), but \( \lim_{s \to \infty} \frac{|g(s, t)|}{t} = 0 \). Thus under assumption (2.4), (3.40) holds for \( i = 3 \).

That (3.40) holds for \( i = 5 \) is immediately checked. Thus, given \( \epsilon > 0 \), there is an \( s(\epsilon) \) such that \( \sup_{s \geq s(\epsilon), 0 \leq t \leq 1} |g_i(s, t)| < \epsilon \) for \( i = 1, 3, 5 \). Also Lemma 3.5 and (3.30) we have,

\[
\int_s^\infty \varphi_1'(\sigma)\varphi_1(\sigma) d\sigma = -\frac{1}{2} \varphi_1^2(s) .
\]

Therefore, for \( s \geq s(\epsilon) \),

\[
\frac{|K(s)|}{\varphi_1^2(s)} \leq C\epsilon \frac{\int_s^\infty \varphi_1(\sigma)\varphi_1'(\sigma) d\sigma}{\varphi_1^2(s)} \leq C\epsilon
\]

and \( \lim_{s \to \infty} \frac{K(s)}{\varphi_1^2(s)} = 0 \) follows.
As for \( \frac{L(s)}{\varphi_1(s)} \), the integration by parts gives that

\[
L(s) = \int_s^\infty \int_0^1 (g_2(s, t) + g_4(s, t)) \frac{\partial v}{\partial \sigma}(s, t) \varphi'_1(\sigma) dt \, d\sigma
\]

\[
= -\int_0^1 (g_2(s, t) + g_4(s, t)) v(s, t) \varphi'_1(s) dt
\]

\[
- \int_s^\infty \int_0^1 \left( \frac{\partial g_2}{\partial \sigma}(s, t) + \frac{\partial g_4}{\partial \sigma}(s, t) \right) v(\sigma, t) \varphi'_1(\sigma) dt \, d\sigma
\]

\[
- \int_s^\infty \int_0^1 ((g_2(s, t) + g_4(s, t)) v(\sigma, t) (\lambda_1 \varphi_1(\sigma)
\]

\[
- \int_0^1 (g_1(\sigma, \tau) + g_3(\sigma, \tau) + g_5(\sigma, \tau)) v(\sigma, \tau) d\tau
\]

\[
- \int_0^1 (g_2(\sigma, \tau) + g_4(\sigma, \tau)) \frac{\partial v}{\partial \sigma}(\sigma, \tau) d\tau) \] \, d\sigma \right) d\sigma dt , \text{ by (3.30)}
\]

\[
(3.44)
\]

\[
= -\int_0^1 (g_2(s, t) + g_4(s, t)) v(s, t) \varphi'_1(s) dt
\]

\[
- \int_s^\infty \int_0^1 \left( \frac{\partial g_2}{\partial \sigma}(s, t) + \frac{\partial g_4}{\partial \sigma}(s, t) \right) v(\sigma, t) \varphi'_1(\sigma) dt \, d\sigma
\]

\[
- \int_s^\infty \int_0^1 (g_2(s, t) + g_4(s, t)) v(\sigma, t) (\lambda_1 \varphi_1(\sigma)
\]

\[
- \int_0^1 (g_1(\sigma, \tau) + g_3(\sigma, \tau) + g_5(\sigma, \tau)) v(\sigma, \tau) d\tau
\]

\[
- \int_0^1 (g_2(\sigma, \tau) + g_4(\sigma, \tau)) \frac{\partial v}{\partial \sigma}(\sigma, \tau) d\tau) \] \, d\sigma \right) d\sigma dt , \text{ by (3.30)}
\]

\[
= I_1(s) + I_2(s) + I_3(s) + I_4(s) + I_5(s) .
\]

These terms are controlled provided we obtain appropriate bounds on \( g_2(\sigma, t), g_4(\sigma, t), \)

\( \frac{\partial g_2}{\partial \sigma}(\sigma, t) \) and \( \frac{\partial g_4}{\partial \sigma}(\sigma, t) \).

For \( g_2(\sigma, t) \) and \( \frac{\partial g_2}{\partial \sigma}(\sigma, t) \), it suffices to handle \( \tilde{c}_2(\sigma, t), \tilde{c}_2(\sigma, t) t^{d-3} \) (possibly troublesome when \( d = 2), \frac{\partial \tilde{c}_2}{\partial t}(\sigma, t), \) and \( \frac{\partial^2 \tilde{c}_2}{\partial \sigma \partial t}(\sigma, t) \).
A quick inspection of $\tilde{c}_2(\sigma, t)$, $\tilde{c}_2(\sigma, t)t^{d-3}$ and
\[
\frac{\partial \tilde{c}_2}{\partial t}(\sigma, t) = \frac{3}{t} \tilde{c}_2(\sigma, t) + \frac{\frac{t^3(5f'^2f'' - ff'^3)}{1 + \ell}}{t^3} x_t + \tilde{c}_2(\sigma, t) \frac{\ell_t}{1 + \ell}
\]
keeping in mind that
\[
(3.45)
\begin{cases}
\ell_t(\sigma, t) = \frac{2t^2k_t}{t^2} - 4t^3 k f'^2 - t^4 k f'^2 - 2t^4 k f'' f_x t \\
+ \frac{t^3 f'^3 f''}{1 + \ell} + t^3 f'^3 f'' f_x t + \frac{t^3 f'^2 f'' f_x t}{1 + \ell}
\end{cases}
\]

One sees that under assumption (2.4)
\[
\lim_{s \to \infty} k_t(s, t) = 0
\]
and
\[
\lim_{s \to \infty} \tilde{c}_2(s, t) \ell_t(s, t) = 0.
\]
Thus, from (3.12), (3.20), (3.45) and assumption (2.4) we get
\[
\lim_{s \to \infty} \sup_{0 \leq t \leq 1} |g_2(s, t)| = 0.
\]
Also,
\[
\frac{\partial^2 \tilde{c}_2}{\partial t \partial s}(\sigma, t) = \frac{3}{t} \left[ \frac{t^3(5f'^2f'' - ff'^3)}{1 + \ell} x_s - \frac{\ell_s(\sigma, t)}{1 + \ell} \right] + \frac{t^3(5f'^2f'' - ff'^3)}{1 + \ell} x_s x_t + \frac{t^3(5f'^2f'' - ff'^3)}{1 + \ell} x_s x_t + \frac{\ell_t(\sigma, t) \ell_s}{1 + \ell} + \frac{\ell_t(\sigma, t) \ell_s}{(1 + \ell)^2},
\]
with
\[
\frac{\partial \tilde{c}_2}{\partial s}(\sigma, t) = \frac{t^3(5f'^2f'' - ff'^3)}{1 + \ell} x_s - \frac{\ell_s(\sigma, t)}{1 + \ell}
\]
21
and from (3.17) and (3.18)

\[(3.47) \quad \ell(s, t) = t^2k_s(1 - t^2f'^2) - 2t^4kf''x_s + \frac{1}{2}t^4(ff'f''^2 + f^2f'f^{(3)})x_s\]

\[(3.48) \quad k_s(\sigma, t) = (ff^{(3)} - ff'')x_s\]

\[(3.49) \quad \ell_{st} = 2tk_s + t^2k_{st} - 4t^3k_sf'^2 - 8t^3kf'f''x_s - 4t^4k_{st}f'^2 - 2t^4ksf''x_s - 2t^4k'f''x_sx_t - 2t^4k'f^{(3)}x_sx_t - 2t^3k'f''x_s + 2t^3(ff'f''^2 + f^2f'^{2})x_s\]

\[+ \frac{1}{2}t^4(ff'^2f''^3) + 4ff'f''f''' + f^2f^{(3)}f''^2 + f^2f''f^{(4)}x_sx_t + \frac{1}{2}t^4(ff'f''^2 + f^2f'^{2})x_{st}\]

\[k_{st} = (ff^{(4)} - f'^{2})x_sx_t + (ff^{(3)} - f'f'')(x_{st})\]

Examining the eight terms in (3.48) in order we see that assumption (2.4) insures that

\[\lim_{s \to \infty} \sup_{0 \leq t \leq 1} \partial^2 \tilde{c}_2(s, t) = 0.\]

As a consequence,

\[\lim_{s \to \infty} \left\{ \sup_{0 \leq t \leq 1} |g_2(s, t)| + \sup_{0 \leq t \leq 1} \left| \frac{\partial g_2}{\partial s}(s, t) \right| \right\} = 0.\]

The verification for \(g_4(s, t)\) and \(\frac{\partial g_4}{\partial s}(s, t)\) is easier. It’s quick to see that

\[\lim_{s \to \infty} \sup_{0 \leq t \leq 1} |g_4(s, t)| = 0\]

follows from (2.4). Also,

\[(3.50) \quad \frac{\partial g_4}{\partial s}(s, t) = \left[3(d + 1)t^2f'^2f'' - \frac{1}{2}(d + 4)t^2[ff'^2f'' + ff'f'^2 + fff^{(3)}]x_s \right] \frac{1}{(1 + \ell)} + \left[\frac{1}{2}t^2[ff'f^{(3)} + f^2f'^{4}]\right]x_s \frac{1}{(1 + \ell)} - c_4 \frac{\ell_s}{1 + \ell},\]
and a routine check gives
\[ \lim_{s \to \infty} \sup_{0 \leq t \leq 1} \left| \frac{\partial g_4}{\partial s}(s, t) \right| = 0 . \]

Returning now to (3.43) we find using (3.29), (3.30) and Lemma 3.5 that
\[ \lim_{s \to \infty} (|I_1(s)| + |I_2(s)| + |I_3(s)| + |I_4(s)| + |I_5(s)|) \varphi_1^{-2}(s) = 0 . \]
This completes the proof. \[ \square \]

**Lemma 3.7.** Under assumption (2.4)
\[ \frac{K(s)}{\varphi_1^2(s)} , \frac{L(s)}{\varphi_1^2(s)} \in L^1(s_0, \infty). \]

**Proof.** The proof uses the integrability conditions of (2.4) and follows the lines of the proof of Lemma 3.6.

The difference being we find now expressions like
\[ \int_{s_0}^{\infty} \int_{s}^{\infty} \int_{0}^{1} f^2 |h| v(\sigma, t) \varphi'_1(\sigma) \varphi_1^{-2}(s) dt d\sigma ds \]
with \( f^2 |h| \in L^1(s_0, \infty), \) by the integrability conditions of (2.4). We proceed by observing that (3.39) and Lemma 3.6 imply that (we might need to increase \( s_0 \))
\[ -\sqrt{\frac{\lambda_1}{2}} \geq \frac{\varphi'_1(s)}{\varphi_1(s)} \geq -\sqrt{\frac{3\lambda_1}{2}}, \text{ for } s \geq s_0 . \]
This implies that for \( \sigma > s_0 \)
\[ |\varphi'_1(\sigma)| \varphi_1(\sigma) \int_{s_0}^{\sigma} \frac{ds}{\varphi_1^2(s)} \leq \sqrt{\frac{3\lambda_1}{2}} \varphi_1^2(\sigma) \int_{s_0}^{\sigma} \frac{ds}{\varphi_1^2(s)} \leq \frac{\sqrt{3}}{2} . \]
Thus,
\[
\int_{s_0}^{\infty} \int_{s}^{\infty} \int_{0}^{1} f^2 |h| v(\sigma, t) |\varphi'_1(\sigma)| |\varphi^{-2}_1(s)| dt \, d\sigma \, ds \\
\leq c \int_{s_0}^{\infty} f^2 \sup_{0 \leq t \leq 1} |h| |\varphi_1(\sigma)| |\varphi'_1(\sigma)| \int_{s_0}^{\sigma} \frac{ds}{\varphi^2_1(s)} d\sigma \quad \text{by Proposition 3.3}
\]
\[
\leq c \sqrt{3} \frac{3\lambda_1}{2} \int_{s_0}^{\infty} f^2 \sup_{0 \leq t \leq 1} |h| |\varphi_1(\sigma)| \int_{s_0}^{\sigma} \frac{ds}{\varphi^2_1(s)} d\sigma
\]
\[
\leq c \sqrt{3} \frac{3\lambda_1}{2} \int_{s_0}^{\infty} f^2 \sup_{0 \leq t \leq 1} |h| d\sigma < \infty .
\]

Returning now to (3.39), we can write it in the form
\[
\frac{\varphi'_1(s)}{\varphi_1(s)} = -\sqrt{\lambda_1 + G(s)}
\]
with \( \lim_{s \to \infty} G(s) = 0 \). Thus,
\[
\frac{\varphi'_1(s)}{\varphi_1(s)} = -\sqrt{\lambda_1 + \frac{1}{\sqrt{\xi}} G(s)}
\]
where \( \xi = \xi(s) \) is between \( \lambda_1 \) and \( \lambda_1 + G(s) \). Then, using Lemma 3.7, \( G \in L^1[s_0, \infty) \) so
\[
(3.51) \quad \varphi_1(s) = \varphi_1(s_0) \exp\{-\sqrt{\lambda_1}(s-s_0) + \int_{s_0}^{s} \frac{1}{\sqrt{\xi(\sigma)}} G(\sigma) d\sigma\}.
\]
Since \( |s - \int_{0}^{x} \frac{d\sigma}{f(\sigma)}| = \frac{1}{2} t^2 |f'(x)| \to 0 \) as \( s \to \infty \), it follows immediately from (3.51) that

**Proposition 3.8.** There exists a positive constant \( C = C(s_0) \) such that
\[
C^{-1} \varphi_1(s_0) e^{-\sqrt{\lambda_1} \int_{s_0}^{s} \frac{d\sigma}{f(\sigma)}} \leq \varphi_1(s) \leq C \varphi_1(s_0) e^{-\sqrt{\lambda_1} \int_{s_0}^{s} \frac{d\sigma}{f(\sigma)}} .
\]

We can now provide

**Proof.** (of Theorem 2.3) Follows from (3.10), (3.51) and Lemma 3.7. \( \square \)
4 Applications

We now turn to applications of our results. First, when we take $h \equiv \mu_1$, the first Dirichlet eigenvalue of $\Delta_d$ on $D$ and $\alpha = \beta \equiv 0$, then $\varphi_1$, the first Dirichlet eigenfunction for $\Delta_d$ on $D$, satisfies (2.1). When we take $h = \alpha = \beta \equiv 0$, then solutions to (2.1) are positive harmonic functions in $D$ vanishing on $\partial D \cap \{ (x,y) : x > 0 \}$. Thus we have the following (recall $\tilde{\lambda}_1$ is first Dirichlet eigenvalue of $\Delta_{d-1}$ on $\Omega$, $z = z(x,y)$ as in Section 2.) Define for $t > 0$, $D_t = \{(x,y) \in D : x > t \}$.

**Corollary 4.1.** Let $\Omega$ be as described in Section 2.

1. If $u$ solves (2.1) with $\alpha = \beta = h \equiv 0$, and (2.2) holds with this choice of $\alpha, \beta, h$ then

   $\forall \epsilon > 0$, there are constants $C_1, x_1$, such that

   $$C_1^{-1} f(x) \frac{2-d}{2} \exp \left\{ -\sqrt{\tilde{\lambda}_1 + \epsilon} \int_0^x \frac{dt}{f(t)} \right\} \tilde{w}_1(z) \leq u(x,y) \leq C_1 f(x) \frac{2-d}{2} \exp \left\{ -\sqrt{\tilde{\lambda}_1 - \epsilon} \int_0^x \frac{dt}{f(t)} \right\} \tilde{w}_1(z),$$

   for $(x,y) \in D_{x_1}$.

2. If $\varphi_1$ is the first Dirichlet eigenfunction for $\Delta_d$ on $D$, and (2.2) holds for $\alpha = \beta \equiv 0$, $h \equiv \mu_1$, then $\forall \epsilon > 0$, there are constants $C_2, x_2$ such that

   $$C_2^{-1} f(x) \frac{2-d}{2} \exp \left\{ -\sqrt{\tilde{\lambda}_1 + \epsilon} \int_0^x \frac{dt}{f(t)} \right\} \tilde{w}_1(z) \leq \varphi_1(x,y) \leq C_2 f(x) \frac{2-d}{2} \exp \left\{ -\sqrt{\tilde{\lambda}_1 - \epsilon} \int_0^x \frac{dt}{f(t)} \right\} \tilde{w}_1(z),$$

   for $(x,y) \in D_{x_2}$.

When $D$ is the horn-shaped region based on $\Omega = B(0,1) \subset \mathbb{R}^{d-1}$ and Theorem 2.3 holds, take $\lambda_1$ to be the first Dirichlet eigenvalue for $\Delta_{d-1}$ on $B(0,1)$ and $w_1$ the corresponding eigenfunction, then
Corollary 4.2. If \( D = \{(x, y) : x > 0, \|y\| < f(x)\} \) and \( f \) satisfies (2.4), then

(1) If \( \Delta_d u = 0, u > 0 \) on \( D \), \( u = 0 \) on \( \partial D \cap \{(x, y) : x > 0\} \) and \( \lim_{x \to \infty} u(x, 0) = 0 \), then there are constants \( C_1, x_1 \) such that

\[
C_1^{-1} f(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)} \right\} w_1 \left( \frac{\|y\|}{f(x)} \right) \leq u(x, y)
\]

\[
\leq C_1 f(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)} \right\} w_1 \left( \frac{\|y\|}{f(x)} \right),
\]

for \( (x, y) \in D_{x_1} \).

(2) If \( \varphi_1 \) is the first Dirichlet eigenfunction of \( \Delta_d \) on \( D \) there are constants \( C_2 \) and \( x_2 \) such that

\[
C_2^{-1} f(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)} \right\} w_1 \left( \frac{\|y\|}{f(x)} \right) \leq \varphi_1(x, y)
\]

\[
\leq C_2 f(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)} \right\} w_1 \left( \frac{\|y\|}{f(x)} \right),
\]

for \( (x, y) \in D_{x_2} \).

These estimates also give information when the operator \( L = \Delta_d + \alpha \frac{\partial}{\partial x} + \beta \nabla_y + h \) corresponds to an intrinsically ultracontractive (IU) semigroup. The reader is referred to the article of Davies and Simon [5] on this subject. The semigroup \( e^{-tL} \) is (IU) if \( \varphi_1^{-1} e^{-t(L-\mu_1)} \varphi_1 \) maps \( L^2(D, \varphi_1^2(x)dx) \) into \( L^\infty(D, \varphi_1^2(x)dx) \) for all \( t > 0 \).

As outlined in [5] whether \( e^{-tL} \) is (IU) depends on the behavior of \( \varphi_1 \). A sufficient condition for \( e^{-tL} \) to be (IU) in our context (see sections 7, 8 and 9 of [5]) is

\[
-\log \varphi_1 \leq \epsilon f^{-2} + e^{\epsilon a}, \quad \text{for some } a > 0, \quad \text{for all small } \epsilon.
\]

(4.1)

In the case of \( L \) satisfying the conditions of (2.2), by Theorem 2.1, (4.1) holds if

\[
\sqrt{\lambda_1 + \delta} \int_0^x \frac{d\tau}{f(\tau)} \leq \epsilon f(x)^{-2} + e^{\epsilon a}, \quad \text{for some } a > 0, \quad \text{for all small } \epsilon.
\]

(4.2)
So, for example, (4.2) holds if \( f(x) = (x + 1)^{-s} (\ell n(x + 2))^{-t} \) when \( s = 1 \) and \( t > 1 \) or \( s > 1 \), as was pointed out in [5]. A few of the consequences of Theorem 2.1 and (4.2) are outlined in the following corollary. These are consequences of \( e^{-tL} \) being IU and the reader is referred to [5] for details and for numerous other consequences.

**Corollary 4.3.** Assume (2.3) with \( \alpha = \beta \equiv 0 \) and (4.2) hold. Let \( \varphi_n \) be the \( n \)th Dirichlet eigenfunction and \( p(t, \cdot , \cdot ) \) the heat kernel for \( \Delta_d + h \) on \( D \). Then there are \( C_0(t), C_n, n \geq 2 \) such that

1. \( C_0^{-1}(t) \exp \left\{ -\sqrt{\lambda_1 + \delta} \left[ \int_0^x \frac{f_0(x)}{f(x)} \, d\tau + \int_0^s \frac{f_0(s)}{f(s)} \, d\tau \right] \right\} w_1(z)w_1(u) \leq p(t, (x, z), (s, u)) \leq C_0(t) \exp \left\{ -\sqrt{\lambda_1 - \delta} \left[ \int_0^x \frac{f_0(x)}{f(x)} \, d\tau + \int_0^s \frac{f_0(s)}{f(s)} \, d\tau \right] \right\} w_1(z)w_1(u) \), for \( (x, z), (s, u) \in D \).

2. \( |\varphi_n(x, z)| \leq C_n \exp \left\{ -\sqrt{\lambda_1 - \delta} \int_0^x \frac{f_0(x)}{f(x)} \, d\tau \right\} w_1(z) \).

In the case when, \( \Omega = B(0, 1) \) we replace (4.2) by

\[
\sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)} \leq e f(x)^{-2} + e^{-a} \quad \text{for some } a > 0, \text{ all small } \epsilon > 0.
\]

Then we obtain

**Corollary 4.4.** Assume (2.4) with \( \alpha = \beta \equiv 0 \) and (4.3) hold. Let \( \varphi_n \) be the \( n \)th Dirichlet eigenfunction and \( p(t, \cdot , \cdot ) \) the heat kernel for \( \Delta_d + h \) on \( D \). Then there are constants \( C_0(t), C_n, n \geq 2 \) such that

1. \( C_0^{-1}(t) \frac{2^{-d}}{2^{-d}} f(x)^{2^{-d}} \exp \left\{ -\sqrt{\lambda_1 [\int_0^x \frac{f_0(x)}{f(x)} \, d\tau + \int_0^s \frac{f_0(s)}{f(s)} \, d\tau ]} \right\} w_1(z)w_1(u) \leq p(t, (x, z), (s, u)) \leq C_0(t) \frac{2^{-d}}{2^{-d}} f(x)^{2^{-d}} \exp \left\{ -\sqrt{\lambda_1 [\int_0^x \frac{f_0(x)}{f(x)} \, d\tau + \int_0^s \frac{f_0(s)}{f(s)} \, d\tau ]} \right\} w_1(z)w_1(u) \), for \( (x, z), (s, u) \in D \).

2. \( |\varphi_n(x, z)| \leq C_n f(x)^{2^{-d}} \exp \left\{ -\sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)} \right\} w_1(z), (x, z) \in D. \)
Finally, we apply our results to bounded domains with cusps. Now let \( f : [0, \infty) \to (0, \infty) \) and define \( D = \{(x, y) : x^2 + \|y\|^2 > 1, x > 0, \|y\| < f(x) \} \). Using the Kelvin transformation we can transfer the results of Section 2 to the present setting. For \( \xi \in \mathbb{R}^d, \xi = (x, y) \), put \( \xi' = \frac{\xi}{\|\xi\|^2}, \xi' = (x', y') \), this is inversion in \( S^{d-1} \). Given \( D, D' = \{\xi' : \xi \in D \} \) has a cusp at the origin we can now see how harmonic functions on \( D' \) which vanish in a neighborhood of the origin on \( \partial D' \) and the first eigenfunction decay in the cusp. Let \( u' \in C^2(D') \), then \( u \in C^2(D) \) where \( u'(\xi') = u(\xi) \) and

\[
\Delta_d' u'(\xi') = \|\xi\|^d \Delta_d \left( \frac{u(\xi)}{\|\xi\|^{d-2}} \right)
\]

where \( \Delta_d' \) denotes the Laplacian in the \( \xi' \)-variable. Let \( \mu_1 \) be the first Dirichlet eigenvalue for \( D \). Thus \( \Delta_d u'_1 + \mu_1 u'_1 = 0 \) in \( D' \) if and only if \( \Delta_d \left( \frac{u_1(\xi)}{\|\xi\|^{d-2}} \right) + (\|\xi\|^{-4} \mu_1) \frac{u_1(\xi)}{\|\xi\|^{d-2}} = 0 \) in \( D \) and, of course, \( u' \) is harmonic on \( D' \) if and only if \( \frac{u(\xi)}{\|\xi\|^{d-2}} \) is harmonic on \( D \). Transferring Corollary 4.2 to the present situation yields

**Theorem 4.5.** Suppose \( f(x) \) satisfies (2.4) with \( h = (x+1)^{-4} \) and let \( D = \{(x, y) : x^2 + \|y\|^2 > 1, x > 0, \|y\| < f(x) \} \). The image of \( D \) under the Kelvin transform with respect to the unit ball in \( \mathbb{R}^d \) will be denoted \( D' \). Then \( D' = \{(x', y') : (x')^2 + \|y'\|^2 < 1, x' > 0, \frac{\|y'\|}{(x')^2 + \|y'\|^2} < f \left( \frac{x'}{(x')^2 + \|y'\|^2} \right) \} \).

1. If \( u' \) is harmonic on \( D' \); \( u' > 0 \) on \( \partial D' \cap \{(x', y') : (x')^2 + \|y'\|^2 < 1 \} \), then there exists \( C \) such that with \( \xi' = (x', y') \),

   (a) \( C^{-1}(\|\xi'\|^2 f \left( \frac{x'}{\|\xi'\|^2} \right))^\frac{2-d}{2} \exp \left\{ -\sqrt{\lambda_1} \int_0^{\|\xi'\|^2} \frac{dr}{f(r)} \right\} w_1 \left( \frac{\|y'\|}{\|\xi'\|^2 f \left( \frac{x'}{\|\xi'\|^2} \right)} \right) \leq u'(x', y') \leq C \left( \|\xi'\|^2 f \left( \frac{x'}{\|\xi'\|^2} \right)^\frac{2-d}{2} \exp \left\{ -\sqrt{\lambda_1} \int_0^{\|\xi'\|^2} \frac{dr}{f(r)} \right\} w_1 \left( \frac{\|y'\|}{\|\xi'\|^2 f \left( \frac{x'}{\|\xi'\|^2} \right)} \right) \right. \). In particular, when \( y' = 0 \),

   (b) \( C^{-1}(x')^2 f \left( \frac{1}{x'} \right)^\frac{2-d}{2} \exp \left\{ -\sqrt{\lambda_1} \int_0^{1/x'} \frac{dr}{f(r)} \right\} w_1(0) \leq u'(x', 0) \leq C(x')^2 f \left( \frac{1}{x'} \right)^\frac{2-d}{2} \exp \left\{ -\sqrt{\lambda_1} \int_0^{1/x'} \frac{dr}{f(r)} \right\} w_1(0) \).
If $\varphi'_1$ is the first Dirichlet eigenfunction on $D'$ with corresponding eigenvalue $\mu_1$, then

1(a) and 1(b) hold with $\varphi'_1$ in place of $u'$.

So, for example, Theorem 4.5 holds if $f(x') = x'((2+\alpha)(-\ell n(x'))^{-\beta}$ when $\alpha = 1$ and $\beta > 1$
or $\alpha > 1$, with domains $D' = \{(x', y') : x'^2 + \|y'\|^2 < 1, x' > 0, \|y'\| < f(x')\}$.

5 Appendix

In this section we shall give some general comparison property and monotone property of positive solutions of (2.1) and (2.3). Let $D, \tilde{w}_1$ and $\tilde{\lambda}_1$ be as in Section 2. We will use $\tilde{D}_x$ for a fixed $x > 0$ to denote the cross section of $D$ at $x$, i.e. the set of $y$ such that $(x, y) \in D$.

Lemma 5.1. If $u$ solves (2.3) and $u(0, y) = u(0, \|y\|)$ is nonincreasing in $\|y\|$ with $h(x, y) = h(x, \|y\|)$ nonincreasing in $\|y\|$, then

$u(x, y) = u(x, \|y\|)$ and $u(x, \|y\|)$ is decreasing in $\|y\|$ for $(x, y) \in D$.

Proof. This result can be proved by the standard moving plane argument used in [6]. Thus we omit its proof here.

Lemma 5.2. If $u$ is a subsolution of (2.1) and $v$ is a supersolution of (2.1) where $\alpha, \beta, h$ and $R, S$ satisfy (2.2), then there exists $x_0 > 0$ such that for $C = \sup_{D_{x_0}} \frac{w(x_0, y)}{v(x_0, y)}$

$u(x, y) \leq Cv(x, y)$ for $(x, y) \in D \cap \{(x, y) : x > x_0\}$.

Proof. Since $v$ is a supersolution of (2.1) and $v = 0$ on $\partial D \cap \{(x, y) : x > 0\}$ the outer normal derivative of $v$ on $\partial D \cap \{(x, y) : x > 0\}$ is negative everywhere. Hence $C$ is well-defined for all $x_0 > 0$. For any $M > 0$, $u - Mv$ is a subsolution of (2.1) in $D$. Then

$w\left(\int_0^x \frac{d\tau}{f(\tau)} \frac{R(x)(y - S(x))}{f(x)}\right) = (u - Mv)(x, y)$.
satisfies for \((s, z) \in (0, \infty) \times \Omega\) with \(s = \int_0^z \frac{dr}{f(r)}\) and \(z = \frac{R(x)(y-S(x))}{f(x)}\) that if \(D\) denotes the derivative with respect to \(\cdot\), then

\[
Aw = \Delta_d w + D^2_z w(f_D x z, f D x z) + \langle \nabla^2_{zz} w, f D x z \rangle + (\alpha f - f')D_s w
\]

\[
+ \langle \nabla_2 w, f^2 D^2 x z \rangle + \alpha f \langle \nabla_2 w, f D x z \rangle + \langle f \beta, \nabla w f D y z \rangle + f^2 h w \leq 0.
\]

Fix \(R\) large enough so that \(\bar{\Omega} \in B(0, R) \subset \mathbb{R}^{d-1}\) and let \(w_R(y)\) and \(\lambda_1(R)\) be the first Dirichlet eigenfunction and eigenvalue for \(\Delta_d\) over \(B(0, R)\). Then \(\frac{w}{w_R}\) satisfies

\[
\begin{align*}
\Delta_d \left( \frac{w}{w_R} \right) &+ D^2_z \left( \frac{w}{w_R} \right)(f D x z, f D x z) + \langle \nabla^2_{zz} \left( \frac{w}{w_R} \right), f D x z \rangle + (\alpha f - f')D_s \left( \frac{w}{w_R} \right) \\
&+ \langle \nabla_2 \left( \frac{w}{w_R} \right), f^2 D^2 x z \rangle + \alpha f \langle \nabla_2 \left( \frac{w}{w_R} \right), f D x z \rangle + \langle f \beta, \nabla \left( \frac{w}{w_R} \right) f D y z \rangle \\
&+ \langle D \left( \frac{w}{w_R} \right), D w_R \rangle + \langle D_z \left( \frac{w}{w_R} \right), D w_{RR} \rangle(f D x z, f D x z) \\
&+ \langle f \beta, \nabla w_R f D y z \rangle \left( \frac{w}{w_R} \right) \leq 0, \\
\frac{w}{w_R} &\neq 0 \text{ on } (0, \infty) \times \partial \Omega.
\end{align*}
\]

From (2.2) it is clear that there exists \(s_0 > 0\) such that in \((s_0, \infty) \times \Omega\)

\[
f^2 h + D^2_z w_R(f D x z, f D x z) + \langle \nabla_2 w_R, f^2 D^2 x z \rangle + \alpha f \langle \nabla_2 w_R, f D x z \rangle + \langle f \beta, \nabla w_R f D y z \rangle \leq \lambda_1(R).
\]

Therefore the maximum principle applies here to yield that

\[
\frac{w}{w_R} \leq \max \{ \sup_{\Omega} \left\{ \frac{w(s_0, z)}{w_R(s_0, z)} \right\}, 0 \} \text{ for } (s, z) \in (s_0, \infty) \times \Omega.
\]

Finally taking \(x_0 > 0\) with \(s_0 = \int_{x_0}^{x_0} \frac{dr}{f(r)}\) and \(M = C\) completes the proof.

\[\square\]

**Lemma 5.3.** Let \(\alpha, \beta, h\) and \(R, S\) satisfy (2.2) in \(D\). Then there exists some \(x_1 \geq x_0\) such that if \(\psi(y)\) is a positive function in \(C^1(\bar{D}_{x_1})\) and \(\psi = 0\) on \(\partial \bar{D}_{x_1}\) with negative outer normal
derivative on $\partial \tilde{D}_{x_1}$, then the following problem

$$
\begin{align*}
&Lu = 0, \text{ on } D \cap \{(x,y) : x > x_1\}, \\
&u > 0, \text{ on } D \cap \{(x,y) : x > x_1\}, \\
&u = 0, \text{ on } \partial D \cap \{(x,y) : x > x_1\}, u(x_1,y) = \psi(y), \text{ on } \tilde{D}_{x_1}, \\
&\lim_{x \to \infty} u(x,y) = 0.
\end{align*}
$$

(5.3)

has a unique solution $u$. Furthermore there exists some constant $C = C(x_1)$ such that for $x > x_1$ we have

$$
C^{-1} \inf_{\tilde{D}_{x_1}} \left\{ \frac{\psi}{\tilde{w}_1} \right\} e^{-\frac{3\sqrt{\lambda_1}}{f} \int_{x_1}^{x} \frac{df}{\tau} \tilde{w}_1(z)} \leq u(x,y) \leq C \sup_{\tilde{D}_{x_1}} \left\{ \frac{\psi}{\tilde{w}_1} \right\} e^{-\frac{\sqrt{\lambda_1}}{f} \int_{x_1}^{x} \frac{df}{\tau} \tilde{w}_1(z)}.
$$

(5.4)

Proof. Fix some $\epsilon < \epsilon_0$ given in Lemma 3.2 so that Proposition 3.1 implies

$$
u_-(x,y) = \exp \left\{ -\frac{3\sqrt{\lambda_1}}{2} \int_{x_1}^{x} \frac{df}{f} \right\} \tilde{w}_1^{-\epsilon}(z)
$$

and

$$
u_+(x,y) = \exp \left\{ -\frac{\sqrt{\lambda_1}}{2} \int_{x_1}^{x} \frac{df}{f} \right\} \tilde{w}_1^{1+\epsilon}(z)
$$

are sub- and super-solutions of (5.3), respectively, for $L$ on $D \cap \{(x,y) : x > x_1\}$ for some $x_1 \geq x_0$.

From the assumptions on $\psi$ we can define

$$
a = \inf_{\tilde{D}_{x_1}} \left\{ \frac{\psi}{\tilde{w}_1^{-\epsilon}} \right\} \text{ and } b = \sup_{\tilde{D}_{x_1}} \left\{ \frac{\psi}{\tilde{w}_1^{1+\epsilon}} \right\}.
$$

(5.5)

Then the monotone iteration method (see Theorem 2.10 in [8]) yields a solution $u$ of (5.3) satisfying

$$
a \exp \left\{ -\frac{3\sqrt{\lambda_1}}{2} \int_{x_1}^{x} \frac{df}{f} \right\} \tilde{w}_1^{-\epsilon}(z) \leq u \leq b \exp \left\{ -\frac{\sqrt{\lambda_1}}{2} \int_{x_1}^{x} \frac{df}{f} \right\} \tilde{w}_1^{1+\epsilon}(z).
$$

(5.6)

Since $x_1 \geq x_0$ Lemma 5.2 implies that such solution is uniqueness. Finally Lemma 3.2, (5.5) and (5.6) implies (5.4), which completes the proof.

□
References


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